Second Year of Computer Engineering (2019 Course)

(210241): Discrete Mathematics

Teaching Scheme	Credit Scheme	Examination Scheme and Marks
Lecture: 03 Hours/Week	03	Mid_Semester (TH): 30 Marks End_Semester (TH): 70 Marks

Marks weightage per unit for examination						
Unit Number	**I**	**II**	**III**	**IV**	**V**	**VI**
Mid_Semester	15	15	-	-	-	-
End_Semester	-	-	18	17	18	17

Prerequisites: Basic Mathematics

Course Objectives

To introduce several Discrete Mathematical Structures found to be serving as tools even today in the development of theoretical computer science.

1. To introduce students to understand, explain, and apply the foundational mathematical concepts at the core of computer science.

2. To understand use of set, function and relation models to understand practical examples, and interpret the associated operations and terminologies in context.

3. To acquire knowledge of logic and proof techniques to expand mathematical maturity.

4. To learn the fundamental counting principle, permutations, and combinations.

5. To study how to model problem using graph and tree.

6. To learn how abstract algebra is used in coding theory.

Course Outcomes

On completion of the course, learner will be able to –

CO1: Formulate problems precisely, solve the problems, apply formal proof techniques, and explain the reasoning clearly.

CO2: Apply appropriate mathematical concepts and skills to solve problems in both familiar and unfamiliar situations including those in real-life contexts.

CO3: Design and analyze real world engineering problems by applying set theory, propositional logic and to construct proofs using mathematical induction.

CO4: Specify, manipulate and apply equivalence relations; construct and use functions and apply these concepts to solve new problems.

CO5: Calculate numbers of possible outcomes using permutations and combinations; to model and analyze computational processes using combinatorics.

CO6: Model and solve computing problem using tree and graph and solve problems using appropriate algorithms.

CO7: Analyze the properties of binary operations, apply abstract algebra in coding theory and evaluate the algebraic structures.

Learning Resources

❖ Text Books:

1. C. L. Liu, “Elements of Discrete Mathematics”‖, TMH, ISBN 10:0-07-066913-9.2.
2. N. Biggs, “Discrete Mathematics”, 3rd Ed, Oxford University Press, ISBN 0 –19-850717–8.

❖ Reference Books:

1. Kenneth H. Rosen, “Discrete Mathematics and its Applications”‖, Tata McGraw-Hill, ISBN 978-0-07-288008-3
2. Bernard Kolman, Robert C. Busby and Sharon Ross, “Discrete Mathematical Structures”‖, Prentice-Hall of India /Pearson, ISBN: 0132078457, 9780132078450.
3. Narsingh Deo, “Graph with application to Engineering and Computer Science”, Prentice Hall of India, 1990, 0 –87692 –145 –4.
4. Eric Gossett, “Discrete Mathematical Structures with Proofs”, Wiley India Ltd, ISBN:978-81-265-2758-8.
5. Sriram P.and Steven S., “Computational Discrete Mathematics”, Cambridge University Press, ISBN 13: 978-0-521-73311-3.

Unit I

Set Theory and Logic

Duration: (07 Hours)

Unit-I: Contents

- **Introduction** and Significance of Discrete Mathematics,
- **Sets–**Naïve Set Theory (Cantorian Set Theory), Axiomatic Set Theory, Set Operations, Cardinality of set, Principle of inclusion and exclusion.
- **Types of Sets**–Bounded and Unbounded Sets, Diagonalization argument, Countable and Uncountable Sets, Finite and Infinite Sets, Countably Infinite and Uncountably Infinite Sets, Power set,
- **Propositional Logic-** Logic, Propositional Equivalences, Application of Propositional Logic-Translating English Sentences,
- Proof by Mathematical Induction and Strong Mathematical Induction
- Exemplar/ Case Studies:

What is Discrete Mathematics?

- **Adjective:** Individually separate and distinct.
- **Synonyms:** separate - detached - distinct - abstract.

- Defining discrete mathematics is **hard** because defining mathematics is hard.

- **What is Mathematics?**

- **Discrete Mathematics** is the study of mathematical structures that are fundamentally discrete rather than continuous.

OR

- **Discrete Mathematics** is the branch of mathematics dealing with objects that can consider only distinct, separated values.

What is Discrete Mathematics?

❖ **Example:**

- Consider the function which gives the number of children of each person reading this.
- What is the range? I'm guessing it is something like **{0, 1, 2, 3}**. Maybe **4** is in there too.
- But certainly there is nobody reading this that has 1.32419 children.
- This output set is discrete because the elements are separate.
- The inputs to the function also form a discrete set because each input is an individual person.

What is SET ???

- A **SET** is an unordered collection of different elements.

- A set can be written explicitly by listing its elements using set bracket.

- If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.

- **Sets** are used to group objects together. Often, but not always, the objects in a set have similar properties.

- We write **$a \in A$** to denote that 'a' is an element of the set A.
- The notation $a \notin A$ denotes that 'a' is not an element of the set A.

- Sets to be denoted -- Uppercase letters.
- Elements of sets -- Lowercase letters.

What is SET ???

❖ Example of Sets::

1. A set of all positive integers.
2. A set of all the planets in the solar system.
3. A set of all the states in India.
4. A set of all the lowercase letters of the alphabet.

❖ Basic Properties of Sets::

1. The change in order of writing the elements does not make any changes in the set.
2. If one or many elements of a set are repeated, the set remains the same.

Some Important Sets::

1. **N** − the set of all natural numbers = {1, 2, 3, 4,.....}
2. **Z** − the set of all integers = {.....,−3,−2,−1,0,1,2,3,.....}
3. **Z+** − the set of all positive integers.
4. **Q** − the set of all rational numbers.
5. **R** − the set of all real numbers.
6. **R+** − the set of positive real numbers.
7. **W** − the set of all whole numbers.
8. **C** − the set of complex numbers.

Set Theory Notation

Notation	Description
{ }	We use these braces to enclose the elements of a set. Eg: A= {1, 2, 3}
:	$\{x : x > 2\}$ is the set of all x **such that** x is greater than 2.
$\in$	$2 \in \{1, 2, 3\}$ asserts that 2 is **an element of** the set {1, 2, 3}.
$\notin$	$4 \notin \{1, 2, 3\}$ because 4 **is not an element of** the set {1, 2, 3}.
$\subseteq$	$A \subseteq B$ asserts that *A* **is a subset of** *B*: every element of *A* is also an element of *B*.
$\subset$	$A \subset B$ asserts that *A* **is a proper subset of** *B*: every element of *A* is also an element of *B*, but *A* , *B*.
$\cap$	$A \cap B$ is the **intersection of** *A* **and** *B*: the set containing all elements which are elements of both *A* and *B*.
$\cup$	$A \cup B$ is the **union of** *A* **and** *B*: is the set containing all elements which are elements of *A* or *B* or both.
$\forall$	**For All** E.g: $\forall x>1, x^2>x$
$\exists$	**There Exists** E.g: $\exists\ x \mid x^2>x$
|A|	The **cardinality (or size) of** *A* is the number of elements in A.

Representation of a Set

- Sets can be represented in two different ways ::
- **Roster or Tabular Form::**

➢ The set is represented by listing all the elements comprising it. The elements are enclosed within braces and separated by commas.

➢ **Example 1**: Set of vowels in English alphabet, A = {a,e,i,o,u}

➢ **Example 2:** Set of odd numbers less than 10, B = {1,3,5,7,9}

- **Set Builder Notation::**

➢ The set is defined by specifying a property that elements of the set have in common.

➢ The set is described as **A={x:p(x)}**

➢ **Example 1**: The set {a,e,i,o,u} is written as − A={x:x is a vowel in English alphabet}

➢ **Example 2**: The set {1,3,5,7,9} is written as − B={x:1≤x<10 and (x%2)≠0}

Cardinality of a Set

❖ **Cardinality of a set S**, denoted by |S|, is the number of elements of the set. The number is also referred as the cardinal number.

❖ If a set has an infinite number of elements, its cardinality is ∞.

➢ **Example 1**: If A= {1,4,3,5} then |A|= 4,

➢ **Example 2:** If B ={1,2,3,4,5,…} then |B| = ∞

➢ **Example 3**: Find the cardinality of C ={23, 24, . . . , 37, 38}.

Solution: |C| =16

➢ **Example 4**: Find the cardinality of D ={1, {2, 3, 4}, ∅}.

Solution: |D| = 3

Type of Set

❖ 1. Finite Set::

- A set which contains a definite number of elements is called a finite set. Empty set is also called a finite set.
- For Example:
  - ✓ The Set of Vowel in English alphabet.
  - ✓ The set of all colors in the rainbow.
  - ✓ $P = \{2, 3, 5, 7, 11, 13, 17, ... 97\}$
  - ✓ $S=\{x \mid x \in N \text{ and } 70 > x > 50\}$
  - ✓ $N = \{x : x \in N, x < 7\}$

Type of Set Contd…

❖ 2. Infinite Set::

- A set which contains infinite number of elements is called an infinite set.
- i.e set containing never-ending elements is called an infinite set.
- For Example:
  - ✓ $A = \{x : x \in N, x > 1\}$
  - ✓ $B = \{x : x \in W, x = 2n\}$
  - ✓ $S = \{x \mid x \in N \text{ and } x > 10\}$
  - ✓ Set of all points in a plane
  - ✓ Set of all prime numbers

Type of Set Contd…

❖ 3. Subset ::

- A set X is a subset of set Y (Written as $X \subseteq Y$) **if and only if** every element of X is also an element of set Y.
- Alternate way to define X is a subset of Y: **$\forall x\ (x \in X) \rightarrow (x \in Y)$**
- Every set is a subset of itself, i.e., $X \subset X$, $Y \subset Y$.
- Empty set is a subset of every set.
- For Example:
  - ✓ Let, Y = {1, 2, 3, 4, 5, 6} and X = {1, 2}. Here set X is a subset of set Y as all the elements of set X is in set Y. Hence, we can write $X \subseteq Y$.
  - ✓ Let, Y = {1, 2, 3} and X = {1, 2, 3}.
  - ✓ **YES** $X \subseteq Y$. As set X is a subset (Not a proper subset) of set Y as all the elements of set X is in set Y.

Type of Set Contd…

❖ 4. Proper Subset::

- The term **"proper subset"** can be defined as "subset of but not equal to". i.e $X \subset Y$ and $X \neq Y$
- A Set X is a proper subset of set Y (Written as $X \subset Y$) if every element of X is an element of set Y and $|X| < |Y|$.
- No set is a proper subset of itself.
- Null set or ∅ is a proper subset of every set.
- For Example:
  - ✓ X={1,2,3} and Y ={1,2,3,4,5} Is: $X \subset Y$?
  - ✓ X = {p, q, r} and Y = {p, q, r, s, t} Is: $X \subset Y$?

Answer: Yes.

Type of Set Contd…

❖ 5. Super Set::

- Whenever a set X is a subset of set Y, we say the Y is a superset of X and written as Y ⊇ X.
- For Example:
  - ✓ X = {a, e, i, o, u} and Y = {a, b, c, ..., z}
- Here X ⊆ Y i.e., X is a subset of Y but Y ⊇ X i.e., Y is a super set of X.

Type of Set Contd…

❖ 6. Universal Set::

- It is a collection of all elements in a particular context or application.
- All the sets in that context or application are essentially subsets of this universal set. Universal sets are represented as U.
- For Example:
  - ✓ We may define U as the set of all animals on earth. In this case, set of all mammals is a subset of U, set of all fishes is a subset of U, set of all insects is a subset of U, and so on.
  - ✓ If **A** = {1, 2, 3}; **B** = {2, 3, 4} ; **C** = {3, 5, 7} then **U** = {1, 2, 3, 4, 5, 7} [Here $A \subseteq U$, $B \subseteq U$, $C \subseteq U$ and $U \supseteq A$, $U \supseteq B$, $U \supseteq C$]
  - ✓ If P is a set of all whole numbers and Q is a set of all negative numbers then the universal set is a set of all integers.

Type of Set Contd…

❖ 7. Empty Set or Null Set::

- A set which does not contain any element is called an empty set, or the null set or the void set and it is denoted by ∅ and is read as phi.
- In roster form, ∅ is denoted by { }.
- An empty set is a finite set, since the number of elements in an empty set is finite, i.e., 0.
- For Example:
  - ✓ $S = \{x \mid x \in N \text{ and } 7 < x < 8\} = \emptyset$

Type of Set Contd…

❖ 8. Singleton Set or Unit Set::

- Singleton set or unit set contains only one element. A singleton set is denoted by {S}.
- For Example:
  - ✓ $S = \{ x \mid x \in N, 7 < x < 9 \} = \{8\}$
  - ✓ Let $A = \{x : x \in N \text{ and } x^2 = 4\}$

    Here A is a singleton set because there is only one element 2 whose square is 4.
  - ✓ Let B = {x : x is a even prime number}

    Here B is a singleton set because there is only one prime number which is even, i.e., 2.

Type of Set Contd…

❖ 9. Equal Set::

- Two sets are equal (=) if and only if they have the same elements.
- Cardinality of equal set is same.
- For Example:
  - ✓ If A={1,2,6}and B={6,1,2}, they are equal as every element of set A is an element of set B and every element of set B is an element of set A.
  - ✓ {1,2,3} = {3,1,2} = {1,2,1,3,2} are equal Sets

    Note: Duplicates don't contribute anything new to a set, so remove them. The order of the elements in a set doesn't contribute anything new.
  - ✓ Are {1,2,3,4} and {1,2,2,4} equal?

  Answer: ---- **NO!**

Type of Set Contd…

❖ 10. Equivalent Set::

- If the cardinalities of two sets are same, they are called equivalent sets.
- Elements need not be equal.
- The symbol for denoting an equivalent set is '↔'.
- For Example:
  - ✓ If A={1,2,6} and B={16,17,22} they are equivalent as cardinality of A is equal to the cardinality of B. i.e. $|A| = |B| = 3$. Therefore $A \leftrightarrow B$.

Type of Set Contd…

❖ 11. Disjoint Set::

- Two sets A and B are said to be disjoint, if they do not have any element in common. OR
- Two sets are called disjoint if their intersection is the empty set.
- Disjoint sets have the following properties:
  - ✓ $n(A \cap B) = \emptyset$
  - ✓ $n(A \cup B) = n(A) + n(B)$
- For Example:
  - ✓ A = {1,2,6} and B={7,9,14}, A = {x : x is a prime number} and B = {x : x is a composite number}.

    Here A and B do not have any element in common and are disjoint sets.
  - ✓ Let A = {1, 3, 5, 7, 9} and B = {2, 4, 6, 8, 10}.

    Because $A \cap B = \emptyset$, A and B are disjoint.

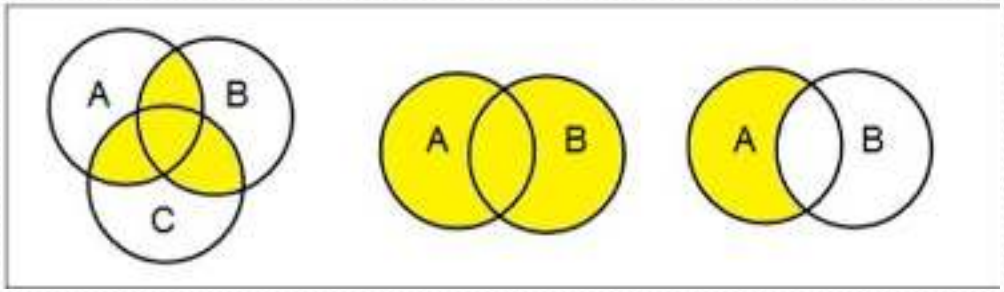
Type of Set Contd…

❖ 12. Overlapping Sets::

- Two sets A and B are said to be overlapping if they contain at least one element in common.
- For Example:
  - ✓ A = {a, b, c, d} and B = {a, e, i, o, u}, Here common element 'a'.
  - ✓ A = {1, 2, 6} and B = {6, 12, 42}. Here common element '6'; hence these sets are overlapping sets.
  - ✓ $X = \{x : x \in N, x < 4\}$ and $Y = \{x : x \in I, -1 < x < 4\}$. Here, the two sets contain three elements in common, i.e., (1, 2, 3).

Venn Diagrams

- Venn diagram, invented in **1880 by John Venn**, is a schematic diagram that shows all possible logical relations between different mathematical sets. OR
- A Venn diagram (**also called primary diagram, set diagram or logic diagram**). is a diagram that shows all possible logical relations between a finite collections of different sets.
- Each circle represents a set. The rectangle containing the circles represents the universe. To represent combinations of these sets, we shade the corresponding region.



Set Operation

❖ **Union Operation::**

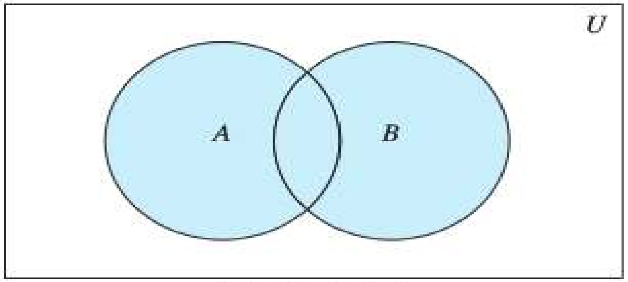
➢ Let A and B be sets. The union of the sets A and B, denoted by **A ∪ B,** is the set that contains those elements that are either in A or in B, or in both.

➢ An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B.

$$A \cup B = \{x \mid x \in A \lor x \in B\}$$

➢ For Example:

✓ If A={10,11,12,13} and B = {13,14,15}, then A∪B={10,11,12,13,14,15}.



A ∪ *B* is shaded.

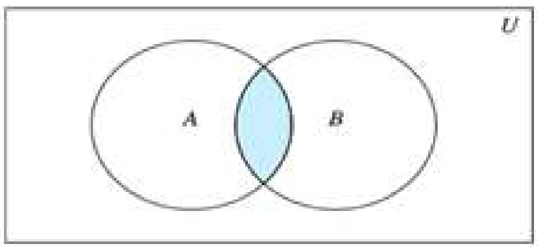
Set Operation

❖ **Interaction Operation::**

- Let A and B be sets. The intersection of the sets A and B, denoted by A ∩ B, is the set containing those elements in both A and B.
- An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B.

$$A \cap B = \{x \mid x \in A \land x \in B\}.$$

- For Example:
  - ✓ If A={10,11,12,13} and B={13,14,15}, then A∩B={13}.



A ∩ B is shaded.

Set Operation

❖ **Difference of Sets (Relative Complement)::**

➢ Let A and B be sets. The difference of A and B, denoted by A − B, is the set containing those elements that are in A but not in B.

➢ The difference of A and B is also called the complement of B with respect to A.

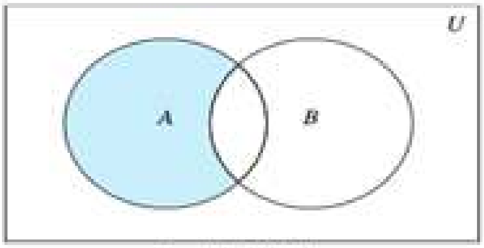
➢ The difference of sets A and B is sometimes denoted by A\B.

➢ An element x belongs to the difference of A & B if & only if $x \in A$ & $x \notin B$.

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

➢ For Example:

✓ If A={1, 3, 5} and B={1, 2, 3} then A-B ={5}



A − B is shaded.

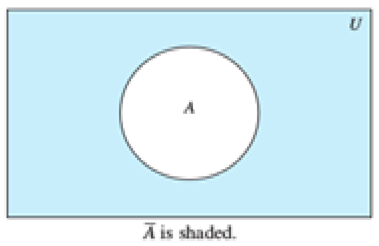
Set Operation

❖ **Complement of a Set::**

➢ Let U be the universal set. The complement of the set A, denoted by Ā, is the complement of A with respect to U. Therefore, the complement of the set A is U − A.

➢ An element belongs to Ā if and only if x ∉ A. This tells us that

$$\bar{A} = \{x \in U \mid x \notin A\}$$



➢ For Example:

✓ Let A = {a, e, i, o, u} and U={a, b, c,…….,z} Then

$$\bar{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}.$$

✓ Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then

$\bar{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Set Operation

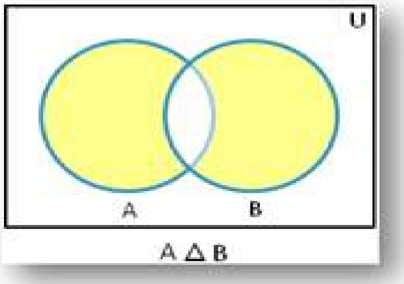
❖ **Symmetric Difference::**

- Let A and B are two sets. The symmetric difference of two sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \triangle B$.

Thus, $A \triangle B = (A - B) \cup (B - A) = \{x : x \notin A \cap B\}$

- or, $A \triangle B = \{x : [x \in A \text{ and } x \notin B] \text{ or } [x \in B \text{ and } x \notin A]\}$.
- $A \triangle B = \{x \mid x \in A - B \vee B - A\}$
- For Example:
- If $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{1, 3, 5, 6, 7, 8, 9\}$, Then

  $A - B = \{2, 4\}$, $B - A = \{9\}$ and $\mathbf{A \triangle B} = \{2, 4, 9\}$.



- If $P = \{a, c, f, m, n\}$ and $Q = \{b, c, m, n, j, k\}$, Then

  $\mathbf{P \triangle Q} = \{a, b, f, j, k\}$

Cartesian Products

- The order of elements in a collection is often important. Because sets are unordered, a different structure is needed to represent ordered collections. This is provided by ordered n-tuples.
- The ordered n-tuple $(a_1, a_2. . . a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, . . . , and an as its n^{th} element.
- Let A and B be sets. The Cartesian product of A and B, denoted by **A × B**, is the set of all ordered pairs (a, b), where a ∈ A and b ∈ B. Hence, **$A \times B = \{(a, b) \mid a \in A \land b \in B\}$**
- Cardinality of the Cartesian product: |A x B| = |A| * |B|.
- For Example:
  - ✓ What is the Cartesian product of A = {1, 2} and B = {a, b, c}?
  - ✓ A × B = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)}.
  - ✓ B × A = {(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)}
  - ✓ Note that the Cartesian products A×B and B×A are not equal, unless A = ∅ or B = ∅.

Power Sets

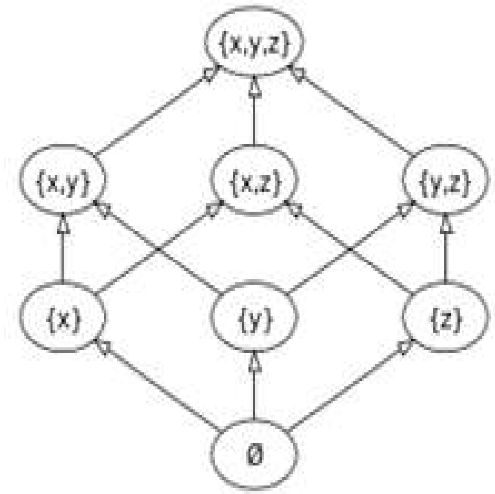
- Given a set S, the power set of S is the set of all subsets of the set S, including the empty set and S itself.
- The power set of S is denoted by P(S).
- If a set has n elements, then its power set has **2^n** elements.
- For Example:

✓ If S is the set {x, y, z}, then the subsets of S are

{}, {x}, {y}, {z}, {x, y}, {x, z}, {y, z}, {x, y, z}

and hence

P(S)={{}, {x}, {y}, {z}, {x, y}, {x, z}, {y, z}, {x, y, z}}.



Power Sets

- For Example:
- ✓ What is the power set of the set {0, 1, 2}?

Solution: **P({0, 1, 2})** ={∅, {0}, {1}, {2}, {0, 1}, {0, 2}, {1, 2}, {0, 1, 2}}.

- ✓ What is the power set of the empty set? And
- ✓ What is the power set of the set {∅}?

Solution: The empty set has exactly one subset, namely, itself. Consequently,

P (∅) = {∅}.

The set {∅} has exactly two subsets, namely, ∅ and the set {∅} itself.

Therefore, **P ({∅}) = {∅, {∅}}.**

Set Identities

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Cardinal Properties of Sets

- If A and B are finite sets, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- If $A \cap B = \phi$, then $n(A \cup B) = n(A) + n(B)$
- $n(A \cap B) = n(A) + n(B) - n(A \cup B)$
- $n(A - B) = n(A) - n(A \cap B)$
- $n(B - A) = n(B) - n(A \cap B)$

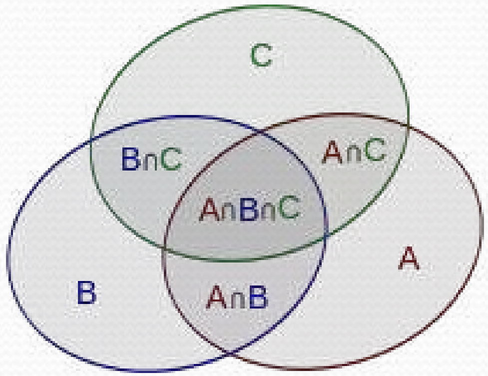
Principle of Inclusion and Exclusion

- The **inclusion–exclusion principle** is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets; symbolically expressed as

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- The principle is more clearly seen in the case of three sets, which for the sets A, B and C is given by

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Principle of Inclusion and Exclusion

❖ **To find the cardinality of the union of n sets:**

- ✓ Include the cardinalities of the sets.
- ✓ Exclude the cardinalities of the pairwise intersections.
- ✓ Include the cardinalities of the triple-wise intersections.
- ✓ Exclude the cardinalities of the quadruple-wise intersections.
- ✓ Include the cardinalities of the quintuple-wise intersections.
- ✓ Continue, until the cardinality of the n-tuple-wise intersection is included (if n is odd) or excluded (n even).

✓ In its general form, the principle of inclusion–exclusion states that for finite sets A1, ..., An, one has the identity:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \cdots$$
$$\cdots + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n| .$$

Example on Principle of Inclusion and Exclusion

❖ **Example 1::** Let A and B be two finite sets such that $n(A) = 20$, $n(B) = 28$ and $n(A \cup B) = 36$, find $n(A \cap B)$.

Solution: Using the formula $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

then $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$$n(A \cap B) = 20 + 28 - 36 = 48 - 36 = 12$$

❖ **Example 2::** If $n(A-B) = 18$, $n(A \cup B) = 70$ & $n(A \cap B)=25$, then find $n(B)$.

Solution: Using the formula $n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$

$$70 = 18 + 25 + n(B - A)$$

$$\therefore n(B - A) = 27$$

Now by using Cardinal Properties of Sets we have,

$$n(B) = n(A \cap B) + n(B - A) = 25 + 27 = 52$$

Example on Principle of Inclusion and Exclusion

❖ **Example 3::** In a group of 60 people, 27 like cold drinks and 42 like hot drinks and each person likes at least one of the two drinks. How many like both coffee and tea?

Solution: Let A = Set of people who like cold drinks.

B = Set of people who like hot drinks.

Given: $(A \cup B) = 60$ $n(A) = 27$ $n(B) = 42$ then;

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

$$= 27 + 42 - 60 = 09$$

Therefore, 9 people like both tea and coffee.

Example on Principle of Inclusion and Exclusion

❖ **Example 4::**In a class of 40 students, 15 like to play cricket and football and 20 like to play cricket. How many like to play football only but not cricket?

Solution: Let C= Students who like cricket & F= Students who like football

$C \cap F$ = Students who like cricket and football both

$C - F$ = Students who like cricket only

$F - C$ = Students who like football only.

Given: $n(C) = 20$ $n(C \cap F) = 15$ $n(C \cup F) = 40$ $n(F) = ?$

$$n(C \cup F) = n(C) + n(F) - n(C \cap F)$$

$$40 = 20 + n(F) - 15$$

$$40 = 5 + n(F)$$

$\therefore n(F) = 35$

$\therefore n(F - C) = n(F) - n(C \cap F) = 35 - 15 = 20$

$\therefore$ Number of students who like football only but not cricket = 20

Example on Principle of Inclusion and Exclusion

❖ **Example 5::** In a survey of university students, 64 had taken mathematics course, 94 had taken chemistry course, 58 had taken physics course, 28 had taken mathematics and physics, 26 had taken mathematics and chemistry, 22 had taken chemistry and physics course, and 14 had taken all the three courses. Find how many had taken one course only.

Solution:

- **Step 1**: Let M, C, P represent sets of students who had taken mathematics, chemistry and physics respectively
- **Step 2**: From the given information, we have $n(M) = 64$, $n(C) = 94$, $n(P) = 58$, $n(M \cap P) = 28$, $n(M \cap C) = 26$, $n(C \cap P) = 22$, $n(M \cap C \cap P) = 14$
- **Step 3**: No. of students who had taken only Math

$$= n(M) - [n(M \cap P) + n(M \cap C) - n(M \cap C \cap P)]$$

$$= 64 - [28+26-14]$$

$$= 64 - 40$$

$$= 24$$

Example on Principle of Inclusion and Exclusion

- **Step 4** : No. of students who had taken only Chemistry

$= n(C) - [n(M \cap C) + n(C \cap P) - n(M \cap C \cap P)]$

$= 94 - [26+22-14]$

$= 94 - 34$

$= 60$

- **Step 5** : No. of students who had taken only Physics

$= n(P) - [n(M \cap P) + n(C \cap P) - n(M \cap C \cap P)]$

$= 58 - [28+22-14]$

$= 58 - 36$

$= 22$

- **Step 6** : Total no. of students who had taken only one course

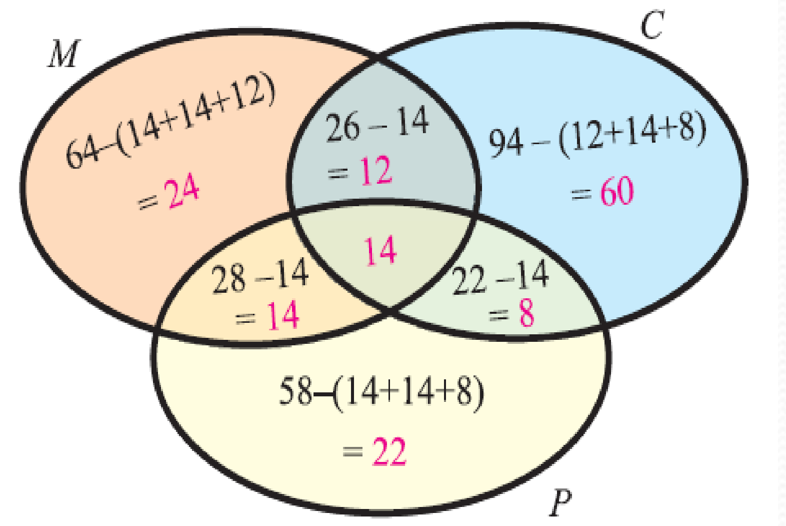
$= 24 + 60 + 22$

$= 106$

Example on Principle of Inclusion and Exclusion

❖ Example 5:: Alternative Method (Using Venn diagram)

- Step 1: Venn diagram related to the information given in the question:



- Step 2: From Venn diagram above, we have

  No. of students who had taken only math = 24

  No. of students who had taken only chemistry = 60

  No. of students who had taken only physics = 22

- Step 3: Total no. of students who had taken only one course = 24 + 60 + 22= 106
- Hence, the total number of students who had taken only one course is 106

Example on Principle of Inclusion and Exclusion

❖ **Example 6::** In a group of students, 65 play foot ball, 45 play hockey, 42 play cricket, 20 play foot ball and hockey, 25 play foot ball and cricket, 15 play hockey and cricket and 8 play all the three games. Find the total number of students in the group. (Assume that each student in the group plays at least one game.)

Solution:

Step 1: Let F, H and C represent the set of students who play foot ball, hockey and cricket respectively.

Step 2: From the given information, we have

$n(F) = 65$, $n(H) = 45$, $n(C) = 42$,

$n(F \cap H) = 20$, $n(F \cap C) = 25$, $n(H \cap C) = 15$ $n(F \cap H \cap C) = 8$

Step 3: Total number of students in the group = $n(F \cup H \cup C)$

$$= n(F) + n(H) + n(C) - n(F \cap H) - n(F \cap C) - n(H \cap C) + n(F \cap H \cap C)$$

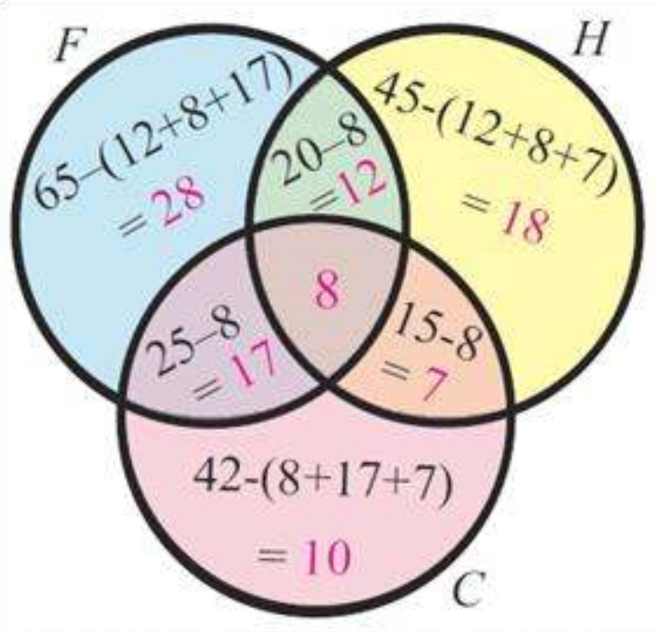
$$= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100$$

Hence, the total number of students in the group is 100

Example on Principle of Inclusion and Exclusion

❖ **Example 6:: Alternative Method (Using Venn diagram)**

➢ **Step 1 :**Venn diagram related to the information given in the question:



➢ **Step 2 :**Total number of students in the group

$$= 28 + 12 + 18 + 7 + 10 + 17 + 8 = 100$$

➢ Hence, the total number of students in the group is 100

Mathematical Induction

- Mathematical Induction is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural number.
- The technique involves two steps to prove a statement, as stated below –
- **Step 1 (Base step)** − It proves that a statement is true for the initial value. **(i. e. $n=n_0$)**
- **Step 2 (Inductive step)** − It proves that if the statement is true for the n^{th} iteration (or number n), then it is also true for $(n+1)^{th}$ iteration (or number n+1).

  (i.e. Statement is true for n=k+1, assuming that it is true for n=k, ($k \geq n_0$))
- **How to Do It:**
- **Step 1** − Consider an initial value for which the statement is true. It is to be shown that the statement is true for n = initial value.
- **Step 2** − Assume the statement is true for any value of n = k. Then prove the statement is true for n = k+1. We actually break n = k+1 into two parts, one part is n = k (which is already proved) and try to prove the other part.

Mathematical Induction

- **Have you heard of the "Domino Effect"?**

  - **Step 1**. The first domino falls
  - **Step 2.** When any domino falls, the next domino falls

So ... all dominos will fall!

That is how Mathematical Induction works.

- **In the world of numbers we say:**
  - ✓ Step 1. Show it is true for first case, usually n=1
  - ✓ Step 2. Show that if n=k is true then n=k+1 is also true

https://www.mathsisfun.com/algebra/mathematical-induction.html

Mathematical Induction Steps

The four steps of math induction:

1. Show $P(1)$ is true

   Let $n = 1$ and work it out.

2. Assume $P(k)$ is true

   Stick a k in for all the n's and say it's true.

3. Show $P(k) \rightarrow P(k+1)$

   * In math, the arrow $\rightarrow$ means "implies" or "leads to."

   USE $P(k)$ to show that $P(k+1)$ is true.

   Very important!

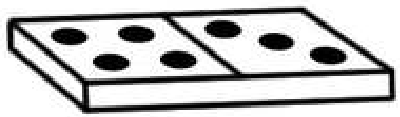
4. End the proof

   Write "Thus, $P(n)$ is true." ■

   This is the modern way to end a proof.

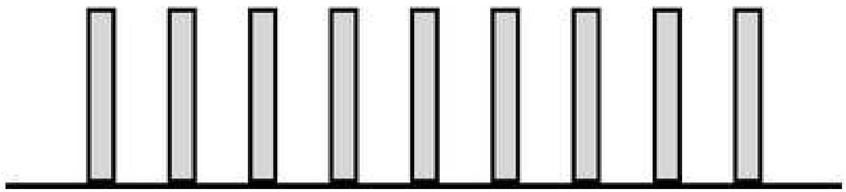
Mathematical Induction

Let's look at some dominoes...



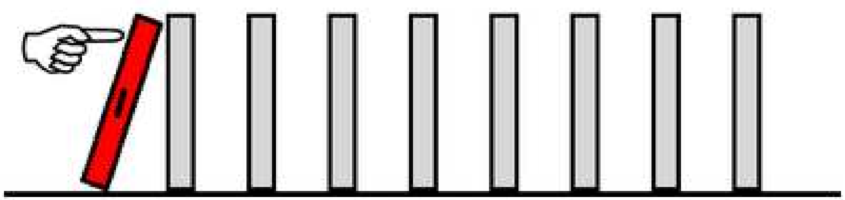
Did you ever stack them so you could knock them all down? It's actually pretty fun and, if you've never done it, I highly recommend that you do.

Let's line up a row of dominoes...



There are four main parts to math induction...

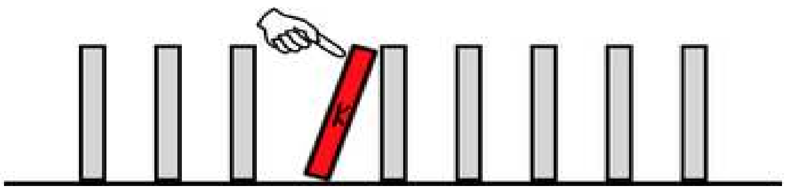
① Can we knock down the first domino?



Yes!

Show $P(1)$ is true.

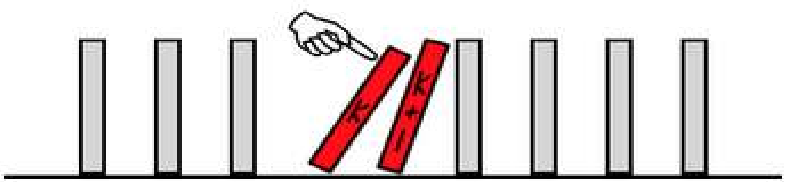
② Can we knock down a random domino somewhere in the middle?
Let's call it the kth domino.



Yes!

Assume $P(k)$ is true.

③ (This one is the big deal.)
If we knock down that kth domino, will the next domino get knocked down too?



Show $P(k) \rightarrow P(k+1)$.

④ If we do all of the above, will all the dominoes fall?

YES!

Thus, $P(n)$ is true.

Mathematical Induction: Example 1:

Prove

$$P(n): 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

① Show $P(1)$ is true:

Stick a 1 in for all the n's and show it works.

$$P(1): 1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

left side ↗ right side ↗

So, $P(1)$ is true.

② Assume $P(k)$ is true:

Stick a k in for all the n's and say it's true.

$$P(k): 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$$

is true.

③ Show $P(k) \rightarrow P(k+1)$

Use $P(k)$ to show that $P(k+1)$ is true.

Write out your goal by sticking $(k+1)$ in for all the n's... Leave the k on the left side.

Mathematical Induction: Example 1:

GOAL: $P(k+1)$:

$$1+2+3+\dots+k+(k+1)=\frac{(k+1)[(k+1)+1]}{2}$$

Start with the left side of the $P(k+1)$ and slowly turn it into how the right side looks.

$P(k+1)$: $\underbrace{1+2+3+\dots+k}_{\text{left side of } P(k)}+(k+1)$

Remember to use $P(k)$!

$$=\underbrace{\frac{k(k+1)}{2}}_{\text{right side of } P(k)}+(k+1)$$

Let your goal guide you!

$$=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}=\frac{k(k+1)+2(k+1)}{2}$$

need a 2 on the bottom

make one fraction

Notice that there is a $(k+1)$ in the front of your goal, so factor that out.

$$=\frac{(k+1)+(k+2)}{2}=\frac{(k+1)[(k+1)+1]}{2}$$

just a little rewrite here

So, $P(k)\rightarrow P(k+1)$

(4) Thus, $P(n)$ is true. ■

Mathematical Induction: Example 2:

Prove

$$P(n):\ 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

① Show $P(1)$ is true:

> Stick a 1 in for all the n's and show it works.

$$P(1):\ 1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = \frac{6}{6} = 1$$

So, $P(1)$ is true.

② Assume $P(k)$ is true:

> Stick a k in for all the n's and say it's true.

$$P(k):\ 1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

is true

③ Show $P(k) \rightarrow P(k+1)$

> Use $P(k)$ to show that $P(k+1)$ is true.

> Write out your goal by sticking $(k+1)$ in for all the n's... Leave the k on the left side.

Mathematical Induction: Example 2:

GOAL: $P(k+1)$:

$P(k+1)$: $1^2+2^2+3^2+\ldots+k^2+(k+1)^2$

$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

Start with the left side of the $P(k+1)$ and slowly turn it into the right side...
Don't confuse the monkey!

$P(k+1)$: $\underbrace{1^2+2^2+3^2+\ldots+k^2}_{P(k)}+(k+1)^2$

Use $P(k)$!

$$= \underbrace{\frac{k(k+1)(2k+1)}{6}}_{P(k)} + (k+1)^2$$

need that 6 and one big fraction

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

Look at the end goal... What's in front? Factor it out!

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

← Simplify this end part...

$$= \frac{(k+1)[2k^2+k+6k+6]}{6}$$

factor this... it MUST work!

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

You can look at the goal to cheat!

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

← now, make these last parts look like the goal

Mathematical Induction: Example 2:

$$= \frac{(k+1)[(k+1)+1][2k+2+1]}{6}$$

$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

So, $P(k) \rightarrow P(k+1)$

4 Thus, $P(n)$ is true. ■

Mathematical Induction: Example 3:

Prove $P(n): \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

① Show $P(1)$ is true:

$$P(1): \left(\frac{a}{b}\right)^1 = \frac{a}{b} = \frac{a^1}{b^1}$$

A picky but important note is that you can't skip this middle step. If you do, then you'd be using the rule you are trying to prove... which is circular reasoning.

So, $P(1)$ is true.

② Assume $P(k)$ is true:

$$P(k): \left(\frac{a}{b}\right)^k = \frac{a^k}{b^k}$$ is true

③ Show $P(k) \rightarrow P(k+1)$

Use $P(k)$ AND $P(1)$ to show that $P(k+1)$ is true.

GOAL: $P(k+1): \left(\frac{a}{b}\right)^{k+1} = \frac{a^{k+1}}{b^{k+1}}$

Start with the left side... Use $P(k)$ and $P(1)$ show every little step... Don't confuse the monkey... End up with the right side.

$$P(k+1): \left(\frac{a}{b}\right)^{k+1} = \underset{P(k)}{\left(\frac{a}{b}\right)^k} \underset{P(1)}{\left(\frac{a}{b}\right)^1} = \underset{P(k)}{\frac{a^k}{b^k}} \cdot \underset{P(1)}{\frac{a^1}{b^1}}$$

$$= \frac{a^k a^1}{b^k b^1} = \frac{a^{k+1}}{b^{k+1}}$$

So, $P(k) \rightarrow P(k+1)$

④ Thus, $P(n)$ is true. ■

Mathematical Induction: Examples

1. With the help of mathematical induction prove that,

$$1^2 + 3^2 + 5^2 + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

2. Prove using mathematical induction that for all $n \geq 1$

$$1 + 4 + 7 + \ldots\ldots + (3n-2) = \frac{n(3n-1)}{2}$$

3. Use the Principle of Mathematical Induction to verify that, for n any positive integer, $6^n - 1$ is divisible by 5.

4. Use the Principle of Mathematical Induction to verify that, for n any positive integer, $3^n - 1$ is a multiple of 2.

Bounded & Unbounded Set

- Let S be a subset of R then,

1. S is said to be **bounded above** if there exists a number M in R such that $\mathbf{x \leq M}$ for every $x \in S$. M is called an **upper bound** for S.
2. S is said to be **bounded below** if there exists a number m in R such that $\mathbf{x \geq m}$ for every $x \in S$. m is called a **lower bound** for S.
3. S is said to be bounded if it is bounded above and below.
4. Sis said to be **unbounded** if it lacks either an upper bound or a lower bound.

Bounded & Unbounded Set Example

- Consider the finite set B = {2,12,0,5,-7,-2} here 12 is upper bounded and -7 is lower bounded. Hence B is bounded.
- The set N of natural number is bounded below but not bounded above.
- The interval [0,1] is bounded.
- Consider set S= {1, 1/2,1/3,1/4……..}. This set consist all numbers of the form 1/n where n ∈ N. We observe that all the number in set S are less than equal to 1 and also observe that no number in set S is less then 0.Thus we say 1 is uppper and 0 is lower bounded respectively for set S.

Bounded & Unbounded Set Example

- Consider sets C={4,6,8,10,……} and D= {0,-1,-2,-3,…..}.
- Each element of C is greater than or equal 4. Hence 4 is lower bound of C and thus C is bounded below. From the nature of the element of C, we note that for any number u, however large, there are always elements of C greater than u. Therefore, u cannot be upper bound of C. Thus C has no upper bound.
- Similarly, it can be seen that the set D is not bounded below although it is bounded above.
- Hence both the sets C and D are unbounded sets.

Countably Infinite and Uncountably Infinite Sets

- A set is **countably infinite** if its elements can be put in one-to-one correspondence with the set of natural numbers.
- In other words, **one can count off all elements** in the set in such a way that, even though the counting will take forever, you will get to any particular element in a finite amount of time.
- **Example:** The integers Z form a countable set.

Countably Infinite and Uncountably Infinite Sets

- A set is **uncountable** if it contains so many elements that they cannot be put in one-to-one correspondence with the set of natural numbers.
- In other words, there is **no way that one can count** off all elements in the set in such a way that, even though the counting will take forever, you will get to any particular element in a finite amount of time. **OR**
- In mathematics, an uncountable set (or uncountably infinite set) is an infinite set that contains too many elements to be countable.
- The **uncountability** of a set is closely related to its **cardinal number**: a set is uncountable if its cardinal number is larger than that of the set of all natural numbers.
- Example of an uncountable set is the set R of all real numbers;

Multiset or bags

- A generalization of the concept of set in which elements may appear **multiple times:** an unordered sequence of elements. OR
- A **multiset** (mset, for short) is an unordered collection of objects (called the elements) in which, unlike a standard (Cantorian) set, elements are allowed to repeat. OR
- In other words, an **mset** is a set to which elements may belong more than once, and hence it is a non-Cantorian set.
- The number of copies of an element appearing in an mset is called its **multiplicity.**

Multiset or bags

- The number of distinct elements in an mset M (which need not be Finite) and their multiplicities jointly determine its cardinality, denoted by C(M).
- In other words, the cardinality of an mset is the **sum of multiplicities of all its elements.**
- An mset M is called Finite if the number of distinct elements in M and their multiplicities are both Finite, it is infinite otherwise.
- For Example: The multisets {a,a,b}, {a,b,a} and {b,a,a} are the same but not equal to either {a,b,b} or to {a,b}.
- Two important Characteristics is of Msets:
  - ✓ There may be repeated occurrences of elements.
  - ✓ There is no particular order or arrangement of the elements.

Multiset or bags

- Two important Characteristics is of Msets:
  - ✓ There may be repeated occurrences of elements.
  - ✓ There is no particular order or arrangement of the elements.
- In fact we can characterize a multiset as a pair of (A, μ),where A is generic set and μ is the multiplicity function defined as

$$\mu: A \rightarrow \{1, 2, 3, \ldots\}$$

- so that μ(a)=k, where k is number of times the element a occur in the mset.
- For Example: if [a, b, c, c, a, c] is the mset, μ(a)=2, μ(b)=1, and μ(c)=3.

Multiset or bags

- **Equality of Multiset:** If the number of occurrences of each element is the same in both the msets, then the msets are equal.
  - ✓ For Example: [a,b,a,a] = [a,a,b,a] and [a,b,a] ≠ [a,b]
- **Multisubset or Msubset:** A multiset A is said to be a multisubset of B if multiplicity of each element in A is less or equal to its multiplicity in B.
  - ✓ For Example: [1,2,2,3] ⊆ [1,1,1,2,2,3]
- **Union and Intersection of Msets:** Let A and B be Msets, and m and n be the number of times x occurs in A and B respectively. Put the larger of m and n occurrences of x in A U B. Put the smaller of m and n occurrences of x in A ∩ B.
  - ✓ For Example: [2, 2, 3] U [2, 3, 3, 4] = [2, 2, 3, 3, 4] & [2, 2, 3] ∩ [2, 3, 3, 4] =[2,3].

Propositional Logic

- A statement can be defined as a **declarative sentence**, or part of a sentence, that is capable of having a truth-value, such as being true or false.
- A propositional consists of propositional variables and connectives.
- The propositional variables are denoted by capital letters (A, B, etc) and connectives connect the propositional variables.
- All the following declarative sentences are propositions:
  1. The sun rises in the East and sets in the West.
  2. Narendra Modi is the 14^{th} Prime Minister of India.
  3. Mumbai is the capital of India.
  4. $1 + 1 = 2$.
  5. $2 + 2 = 3$.
- Propositions 1, 2 and 4 are true, whereas 3 and 5 are false.

Propositional Logic

- Now Consider the following sentences:
  1. What time is it?
  2. Read this carefully
  3. x + 1 = 2.
  4. x + y = z.
  - ✓ Sentences 1 & 2 are not propositions because they are not declarative sentences.
  - ✓ Sentences 3 & 4 are not propositions because they are neither true nor false.
  - ✓ Here sentences 3 & 4 can be turned into a proposition if we assign values to the variables.
- Sometimes, a statement can contain one or more other statements as parts.
- When two statements are joined together with "and", "or", the complex statement formed:
  - ✓ Paris is the capital of France and Paris has a population of over two million.

Propositional Logic

- The Propositional logic and their The truth table is as follows:
  - ✓ Negation/ NOT (¬)
  - ✓ OR (∨) (Disjunction)
  - ✓ AND (∧) (Conjunction)
  - ✓ Exclusive OR (⊕)
  - ✓ Implication / if-then (→) (Implication)
  - ✓ If and only if (⇔) (Bi-conditional or Double Implication)

- **The truth table is as follows:**

A	B	¬A	A ∨ B	A ∧ B	A ⊕ B	A → B	A ⇔ B
T	T	F	T	T	F	T	T
T	F	F	T	F	T	F	F
F	T	T	T	F	T	T	F
F	F	T	F	F	F	T	T

Types of propositions based on Truth values

- **Tautology**: A proposition which is always true, is called a tautology.
- **Contradiction:** A proposition which is always false, is called a contradiction.
- **Contingency**: A proposition that is neither a tautology nor a contradiction is called a contingency.
- **Propositional Equivalences:** Two statements X and Y are logically equivalent if any of the following two conditions hold:
  - ✓ The truth tables of each statement have the same truth values.
  - ✓ The bi-conditional statement $X \Leftrightarrow Y$ is a tautology.

Examples on Propositional Logic

- Prove [(A→B) ∧ A]→B is a tautology
- Prove (A∨B) ∧ [(¬A) ∧ (¬B)] is a contradiction
- Prove (A∨B) ∧ (¬A) a contingency

A	B	A → B	(A → B) ∧ A	[(A → B) ∧ A] → B
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	F	**T**
F	F	T	F	**T**

A	B	A ∨ B	¬ A	¬ B	(¬ A) ∧ (¬ B)	(A ∨ B) ∧ [(¬ A) ∧ (¬ B)]	(A ∨ B) ∧ (¬ A)
T	T	T	F	F	F	**F**	**F**
T	F	T	F	T	F	**F**	**F**
F	T	T	T	F	F	**F**	**T**
F	F	F	T	T	T	**F**	**F**

Examples on Propositional Logic

- **Prove ¬ (A∨ B) and [(¬A) ∧ (¬B)] are propositional equivalent:**
- **Testing by 1st method (Matching truth table):**

A	B	A ∨ B	¬ (A ∨ B)	¬ A	¬ B	[(¬ A) ∧ (¬ B)]
T	T	T	**F**	F	F	**F**
T	F	T	**F**	F	T	**F**
F	T	T	**F**	T	F	**F**
F	F	F	**T**	T	T	**T**

- ✓ Here, we can see the truth values of ¬ (A∨B) and [(¬A) ∧ (¬B)] are same, hence the statements are equivalent.

- **Testing by 2nd method (Bi-conditionality):**

A	B	¬ (A ∨ B)	[(¬ A) ∧ (¬ B)]	[¬ (A ∨ B)] ⇔ [(¬ A) ∧ (¬ B)]
T	T	F	F	T
T	F	F	F	T
F	T	F	F	T
F	F	T	T	T

- ✓ As [¬ (A∨B)] ⇔ [(¬A) ∧ (¬B)] is a tautology, the statements are equivalent.

Inverse, Converse, and Contra-positive

- **Implication / if-then (→) is also called a conditional statement. It has two parts −**
  - ✓ **Hypothesis, P**
  - ✓ **Conclusion, Q**
- **As mentioned earlier, it is denoted as P→Q.**
- Example of Conditional Statement − “If you do your homework, you will not be punished.” Here,
  - ✓ "you do your homework" is the hypothesis, P, and
  - ✓ "you will not be punished" is the conclusion, Q.

Inverse, Converse, and Contra-positive

- **Inverse::**
  - ✓ An inverse of the conditional statement is the negation of both the hypothesis and the conclusion. If the statement is "If P, then Q", the inverse will be "If not P, then not Q".

The Inverse of $P \rightarrow Q$ is $\neg P \rightarrow \neg Q$.

- **Converse::**
  - ✓ The converse of the conditional statement is computed by interchanging the hypothesis and the conclusion. If the statement is "If P, then Q", the converse will be "If Q, then P".

The Converse of $P \rightarrow Q$ is $Q \rightarrow P$.

- **Contra-positive::**
  - ✓ The contra-positive of the conditional is computed by interchanging the hypothesis and the conclusion of the inverse statement. If the statement is "If P, then Q", the contra-positive will be "If not Q, then not P".

The Contra-positive of $P \rightarrow Q$ is $\neg Q \rightarrow \neg P$.

Inverse, Converse, and Contra-positive

- To Summarize,

Statement	IF P then Q	$P \rightarrow Q$
Inverse	IF Not P then NOT Q	$\neg P \rightarrow \neg Q$
Converse	IF Q then P	$Q \rightarrow P$
Contra-positive	IF Not Q then NOT P	$\neg Q \rightarrow \neg P$

- **Example 1:** “If you do your homework , you will not be punished.”
- **Statement:** “If you do your homework , you will not be punished.”
- **Inverse:** “If you do not do your homework, you will be punished.”
- **Converse:** “If you will not be punished, you do your homework”.
- **Contra-positive:** “If you are punished, you did not do your homework”.

Inverse, Converse, and Contra-positive

- **Example 2:** "The home team wins whenever it is raining."
- **Statement:** "If it is raining, then the home team wins."
- **Inverse:** "If it is not raining, then the home team does not win."
- **Converse:** "If the home team wins, then it is raining."
- **Contra-positive:** "If the home team does not win, then it is not raining."

- **Example 3: "**If today is Friday, then it is raining."
- **Statement:** "If today is Friday then it is raining."
- **Inverse: "**If today is not Friday, then it is not raining."
- **Converse: "**If it is raining, then today is Friday."
- **Contra-positive: "**If it is not raining, then today is not Friday."

Logical Equivalences

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Examples

1. Construct the truth table for $(P \rightarrow Q) \wedge (\neg P \Leftrightarrow Q)$
2. Prove that $P \vee \neg P$ is a Tautology
3. Prove that $P \wedge \neg P$ is a Contradiction

Solution 1:

P	Q	¬P	P → Q	¬P ⇔ Q	(P → Q) ∧ (¬P ⇔ Q)
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Solution 2:

P	¬P	P ∨ ¬ P
T	F	T
F	T	T

Solution 3:

P	¬P	P ∧ ¬ P
T	F	F
F	T	F

Examples

4. Determine whether each of the following is a Tautology, a Contradiction or Contingency::

1. $[P \land (P \rightarrow Q)] \rightarrow Q$
2. $(P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P)$
3. $(\neg P \land Q) \land (P \lor \neg Q)$
4. $(P \rightarrow \neg Q) \lor (\neg R \rightarrow P)$
5. $(P \rightarrow Q) \land (\neg P \lor Q)$
6. $(P \rightarrow Q) \rightarrow (P \land Q)$

Solution:

1,2, 4 are Tautology whereas 3 is Contradiction and 5, 6 are Contingency

Examples

5. If $P \rightarrow Q$ is false, determine the value of $(\neg (P \wedge Q)) \rightarrow Q$

6. If P & Q are false , find truth values of $(P \vee Q) \wedge (\neg P \vee \neg Q)$

7. If $P \rightarrow Q$ is true, Can we determine the value $\neg P \vee (P \rightarrow Q)$

P	**Q**	**$P \rightarrow Q$**	**$(P \wedge Q)$**	**$\neg (P \wedge Q)$**	**$(\neg (P \wedge Q)) \rightarrow Q$**
T	F	F	F	T	F

P	**Q**	**$\neg P$**	**$\neg Q$**	**$P \vee Q$**	**$\neg P \vee \neg Q$**	**$(P \vee Q) \wedge (\neg P \vee \neg Q)$**
F	F	T	T	F	T	F

P	**Q**	**$\neg P$**	**$P \rightarrow Q$**	**$\neg P \vee (P \rightarrow Q)$**
T	T	F	T	T
F	T	T	T	T
F	F	T	T	T

Examples

8. For given propositions construct truth tables:

   a) $P \wedge (\neg Q \vee R)$

   b) $(P \wedge Q) \rightarrow \neg R$

   c) $P \rightarrow (\neg Q \vee \neg R)$

   d) $(P \leftrightarrow Q) \leftrightarrow (R \leftrightarrow S)$

   e) $(P \oplus Q) \wedge (P \oplus \neg Q)$

Applications of Propositional Logic

- Logic has many important applications to mathematics, computer science, and numerous other disciplines.
- Statements in mathematics and the sciences in natural language often are imprecise or ambiguous.
- To make such statements precise, they can be translated into the language of logic. For example, logic is used in the specification of software and hardware, because these specifications need to be precise before development begins.
- Furthermore, propositional logic and its rules can be used to design computer circuits, to construct computer programs, to verify the correctness of programs, and to build expert systems.
- Logic can be used to analyze and solve many familiar puzzles. Software systems based on the rules of logic have been developed for constructing some, but not all, types of proofs automatically.

Translating English Sentences

- There are many reasons to translate English sentences into expressions involving propositional variables and logical connectives.
- In particular, English (and every other human language) is often ambiguous.
- Translating sentences into compound statements removes the ambiguity.
- Basic three steps for translation are:
  - ✓ Step 1: Find logical connectives.
  - ✓ Step 2: Break the sentence into elementary propositions.
  - ✓ Step 3: Rewrite the sentence in propositional logic.

Example:

Example 1: You can have free coffee if you are senior citizen and it is a Tuesday

Solution:

Step 1: Find logical connectives.

You can have free coffee **if** you are senior citizen **and** it is a Tuesday

Step 2: Break the sentence into elementary propositions.

A: You can have free coffee

B: You are senior citizen

C: It is a Tuesday

Step 3: Rewrite the sentence in propositional logic.

$(B \wedge C) \rightarrow A$

Example:

Example 2: If you are older than 15 or you are with your parents then you can play roll coaster.

Solution:

Step 1: Find logical connectives.

If you are older than 15 **or** you are with your parents then you can play roll coaster.

Step 2: Break the sentence into elementary propositions.

A= you are older than 15

B= you are with your parents

C= you can play roll coaster

Step 3: Rewrite the sentence in propositional logic.

$$(A \vee B) \rightarrow C$$

Example:

Example 3: Express the specification "The automated reply cannot be sent when the file system is full" using logical connectives.

Solution:

Let P : "The automated reply can be sent" & Q : "The file system is full."

$$Q \rightarrow \neg P$$

Example 4: "You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Solution: Let A, C, and F represent respectively "You can access the internet from campus," "You are a computer science major," and "You are a freshman."

$$A \rightarrow (C \lor \neg F)$$

Example:

Example 5: Assume two elementary statements:

P: you drive over 65 mph; Q: you get a speeding ticket.

Translate each of these sentences to logic

a) You do not drive over 65 mph.

b) You drive over 65 mph, but you don't get a speeding ticket.

c) You will get a speeding ticket if you drive over 65 mph.

d) If you do not drive over 65 mph then you will not get a speeding ticket.

e) Driving over 65 mph is sufficient for getting a speeding ticket.

f) You get a speeding ticket, but you do not drive over 65 mph.

Example:

Example 5: Assume two elementary statements:

P: you drive over 65 mph; Q: you get a speeding ticket.

Translate each of these sentences to logic

a) You do not drive over 65 mph.: **$(\neg P)$**

b) You drive over 65 mph, but you don't get a speeding ticket. : **$(P \land \neg Q)$**

c) You will get a speeding ticket if you drive over 65 mph. : **$(P \rightarrow Q)$**

d) If you do not drive over 65 mph then you will not get a speeding ticket.: **$(\neg P \rightarrow \neg Q)$**

e) Driving over 65 mph is sufficient for getting a speeding ticket. : **$(P \rightarrow Q)$**

f) You get a speeding ticket, but you do not drive over 65 mph. : **$(Q \land \neg P)$**

Example:

Example 6: Let P and Q be the propositions: "The election is decided" and "the votes have been counted" respectively. Express each of the propositions as English sentences:

a) $\neg P$

b) $P \vee Q$

c) $\neg P \wedge Q$

d) $Q \rightarrow P$

e) $\neg P \rightarrow \neg Q$

f) $P \Leftrightarrow Q$

g) $\neg Q \vee (\neg P \wedge Q)$

Example:

Example 6: Let P and Q be the propositions: "The election is decided" and "the votes have been counted" respectively. Express each of the propositions as English sentences:

a) $\neg P$: The election is not (yet) decided.

b) $P \vee Q$: The election is decided or the votes have been counted.

c) $\neg P \wedge Q$:The votes have been counted but the election is not (yet) decided.

d) $Q \rightarrow P$: If the votes have been counted then the election is decided.

e) $\neg P \rightarrow \neg Q$:The election is not decided unless the votes have been counted.

f) $P \Leftrightarrow Q$:The election is decided if and only if the votes been counted.

g) $\neg Q \vee (\neg P \wedge Q)$: The votes have not been counted, or they have been counted by the election is not (yet) decided.

The End