Second Year of Computer Engineering (2019 Course)

(210241): Discrete Mathematics

Teaching Scheme	Credit Scheme	Examination Scheme and Marks
Lecture: 03 Hours/Week	03	Mid_Semester (TH): 30 Marks End_Semester (TH): 70 Marks

Marks weightage per unit for examination						
Unit Number	**I**	**II**	**III**	**IV**	**V**	**VI**
Mid_Semester	15	15	-	-	-	-
End_Semester	-	-	18	17	18	17

Prerequisites: Basic Mathematics

Course Objectives

To introduce several Discrete Mathematical Structures found to be serving as tools even today in the development of theoretical computer science.

1. To introduce students to understand, explain, and apply the foundational mathematical concepts at the core of computer science.

2. To understand use of set, function and relation models to understand practical examples, and interpret the associated operations and terminologies in context.

3. To acquire knowledge of logic and proof techniques to expand mathematical maturity.

4. To learn the fundamental counting principle, permutations, and combinations.

5. To study how to model problem using graph and tree.

6. To learn how abstract algebra is used in coding theory.

Course Outcomes

On completion of the course, learner will be able to –

CO1: Formulate problems precisely, solve the problems, apply formal proof techniques, and explain the reasoning clearly.

CO2: Apply appropriate mathematical concepts and skills to solve problems in both familiar and unfamiliar situations including those in real-life contexts.

CO3: Design and analyze real world engineering problems by applying set theory, propositional logic and to construct proofs using mathematical induction.

CO4: Specify, manipulate and apply equivalence relations; construct and use functions and apply these concepts to solve new problems.

CO5: Calculate numbers of possible outcomes using permutations and combinations; to model and analyze computational processes using combinatorics.

CO6: Model and solve computing problem using tree and graph and solve problems using appropriate algorithms.

CO7: Analyze the properties of binary operations, apply abstract algebra in coding theory and evaluate the algebraic structures.

Learning Resources

❖ **Text Books:**

1. C. L. Liu, "Elements of Discrete Mathematics"‖, TMH, ISBN 10:0-07-066913-9.2.
2. N. Biggs, "Discrete Mathematics", 3rd Ed, Oxford University Press, ISBN 0 –19-850717–8.

❖ **Reference Books:**

1. Kenneth H. Rosen, "Discrete Mathematics and its Applications"‖, Tata McGraw-Hill, ISBN 978-0-07-288008-3
2. Bernard Kolman, Robert C. Busby and Sharon Ross, "Discrete Mathematical Structures"‖, Prentice-Hall of India /Pearson, ISBN: 0132078457, 9780132078450.
3. Narsingh Deo, "Graph with application to Engineering and Computer Science", Prentice Hall of India, 1990, 0 –87692 –145 –4.
4. Eric Gossett, "Discrete Mathematical Structures with Proofs", Wiley India Ltd, ISBN:978-81-265-2758-8.
5. Sriram P.and Steven S., "Computational Discrete Mathematics", Cambridge University Press, ISBN 13: 978-0-521-73311-3.

Unit II

Relations and Functions

Duration: (07 Hours)

Mapping of Course Outcomes: CO2,CO4

Unit-II: Contents

- ❖ **Relations** and their Properties, n-ary relations and their applications, Representing relations,
- ❖ Closures of relations, Equivalence relations, Partial orderings, Partitions,
- ❖ Hasse diagram, Lattices, Chains and Anti-Chains, Transitive closure and Warshall's algorithm.
- ❖ **Functions**-Surjective, Injective and Bijective functions, Identity function, Partial function, Invertible function, Constant function, Inverse functions and Compositions of functions,
- ❖ The Pigeonhole Principle.
- ❖ **Exemplar/ Case Studies:** Know about the great philosophers-Dirichlet

Relation-Introduction

- **Relationships between elements of sets are represented using the structure called a relation, which is just a subset of the Cartesian product of the sets.**
- **A common notion of relation is a type of association that exists between two or more objects.**
- **Example:**
  - ✓ Age – height
  - ✓ Mother – daughter
  - ✓ Student – class
  - ✓ Time – temperature
  - ✓ Person – citizenship

Relation-Introduction

- **Example:**
  - “Is the mother of” is a relation between the set of all females and the set of all people.
    - It consists of all the pairs (person 1, person 2) where person 1 is the mother of person 2.
  - x is the father of y.
  - The number x is greater than the number y.
- **From above example it’s clear that order of object is very important.**

Binary Relation

- **A (binary) relation R between the sets A & B (written as R: A ↔ B) is a subset of the Cartesian product A × B.**

  i.e. $R \subseteq A \times B$

- **If $(x, y) \in R$,** we say **x is related to y,** & denote it by **$x R y$.**
- **If $(x, y) \notin R$,** we say **x is not related to y** & denote it by **$x \not R y$.**

- **Example:** Let A={a, b, c} and B={1, 2, 3}.
  - ✓ Is R = {(a,1),(b,2),(c,3)} a relation from A to B? Yes
  - ✓ Is Q = {(1,a),(2,b)} a relation from A to B? No
  - ✓ Is P = {(a,a),(b,c),(b,a)} a relation from A to A? Yes
  - ✓ Is S = {(a,1),(b,2),(c,2)} a relation from A to B? Yes

Binary Relation

- If there are two sets A and B and Relation from A to B is R(x,y)
- The set A is called the domain of the relation and the set B the codomain
  - ✓ **Domain** = Set of first elements in the Cartesian product**.**

    $\{ x \mid (x,y) \in R \text{ for some } y \text{ in } B \}$

  - ✓ **Range =** Set of second elements in the Cartesian product.

    $\{ y \mid (x,y) \in R \text{ for some } x \text{ in } A \}$.

- **Example:** Let A = {1, 2, 3, 4}. Which ordered pairs are in the relation?
  - ✓ $R = \{(a, b) \mid a < b\}$?
  - ✓ R = {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}
  - ✓ Domain= {1, 2, 3} & Range= {2, 3, 4}

Types of Relation

- The **Empty Relation** between sets A and B, or on E, is the empty set ∅.
  - ✓ Example: If set A = {1, 2, 3} then, one of the void relations can be R = {x, y} where, $|x - y| = 8$. Here $R = \emptyset \subseteq A \times A$

- The **Full Relation** between sets A and B is the set A × B.
  - ✓ Example: Let A={a,b,c} and B={1,2,3} then R={(a,1),(b,2),(c,3)}

- The **Identity Relation** on set A is the set $\{(x, x) \mid x \in A\}$
  - ✓ Example: Let A={1,2,3} then I_A={(1,1),(2,2),(3,3)}

- The Relation R in set A is said to **Universe Relation** if R= A × A
  - ✓ Example: A={a,b,c} then R =A × A={(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)}

Types of Relation

❖ The **Inverse Relation** R' (Converse Relation R^c) of a relation R is defined as – R' or $R^c = \{(b, a) \mid (a, b) \in R\}$. Let R be a relation from set A to set B, then inverse relation R' is from set B to set A.

- ✓ Example: Let A = (1, 2, 3, 4, 5) and
- ✓ R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 4), (3, 3), (4, 4), (5, 5)}
- ✓ R' ={(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (2, 2), (4, 2), (3, 3), (4, 4), (5, 5)}

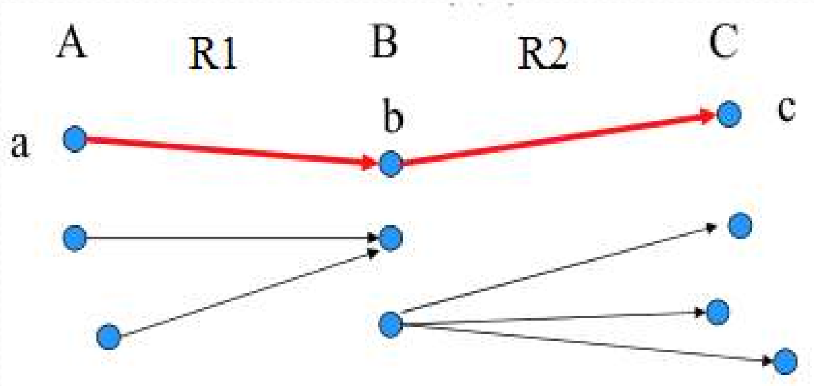
❖ The **Complement of a relation R** is defined as $\hat{R}=\{(a,b) \mid (a,b) \notin R\}$.

❖ i.e. $a \hat{R} b$ iff $a \not R b$ Or $\hat{R} = (A \times B) - R$

- ✓ R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 4), (3, 3), (4, 4), (5, 5)}
- ✓ $\hat{R}$ = {(2, 1), (2, 3), (2, 5), (3, 1), (3, 2), (3, 4), (3, 5),(4, 1), (4, 2), (4, 3), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4)}

Types of Relation

- The **Composite Relation:** Let R_1 be a binary relation from a set A to a set B, R_2 a binary relation from B to a set C.
- Then the composite relation from A to C denoted by R1.R2. Or
- $R1.R2 = \{(a,c) \mid a\, R_1\, b,\ b\, R_2\, c\ ;\ \text{for } a \in A,\ c \in C\}$



- ✓ Example: $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$.
- ✓ $R : A \leftrightarrow B$ $\quad R = \{(1, a), (1, b), (2, b), (2, c)\}$.
- ✓ $S : B \leftrightarrow C$ $\quad S = \{(a, x), (a, y), (b, y), (d, z)\}$.
- ✓ **R.S = {(1, x), (1, y), (2, y)}.**

Types of Relation

- **Combining Relations:** Since relations from A to B are subsets of A x B, two relations from A to B can be combined through set operations.
- **Example:** Let A = {1, 2, 3} & B = {1, 2, 3, 4}. The relations R1 = {(1, 1), (2, 2), (3, 3)} and R2 = {(1, 1), (1, 2), (1, 3), (1, 4)} can be combined to obtain
  - ✓ $R1 \cup R2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$,
  - ✓ $R1 \cap R2 = \{(1, 1)\}$,
  - ✓ $R1 - R2 = \{(2, 2), (3, 3)\}$,
  - ✓ $R2 - R1 = \{(1, 2), (1, 3), (1, 4)\}$,

Example on Types of Relation

❖ **Example 1:** Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.

R = {(1,0), (1,2), (3,1), (3,2)}

S = {(0,b), (1,a), (2,b)} Find R.S

Solution : R.S = {(1,b), (3,a), (3,b)}

❖ **Example 2:** Let R be the relation {(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)}, and let S be the relation {(2, 1), (3, 1), (3, 2), (4, 2)}. Find R.S

Solution: R.S = {(1,1), (1,2), (2,1) (2, 2)}

Example on Types of Relation

❖ **Example 3:** Let A = (1, 2, 3, 4, 5) and R = {(1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,4), (3,3), (4,4), (5,5)}. Find Inverse, Complement, Identity and Universal Relation R.

Solution:

U = {(1,1), (1,2), (1,3), (1,4), (1,5), (2,1),(2,2), (2,3),(2,4), (2,5), (3,1), (3,2), (3,3), (3,4), (3,5), (4,1), (4,2), (4,3), (4,4), (4,5), (5,1), (5,2), (5,3), (5,4), (5,5)}

I_A = {(1,1), (2,2), (3,3), (4,4), (5,5)}

$\hat{R}$ = {(2,1), (2,3), (2,5), (3,1), (3,2), (3,4), (3,5), (4,1), (4,2), (4,3), (4,5), (5,1), (5,2), (5,3), (5,4)}

R' = {(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (2, 2), (4, 2),(3, 3), (4, 4), (5, 5)}.

Example on Types of Relation

❖ **Example 4:** Let A = (1,2,3,4), Let R1 = {(x,y) | x + y = 5} and

R2 = {(x,y)| y - x = 1}. Verify $(R1.R2)^c = R2^c. R1^c$.

Solution:

R1 = {(1,4),(2,3),(3,2),(4,1)} And

R2 = {(1,2),(2,3),(3,4)}

(R1.R2) = {(2,4),(3,3),(4,2)}

$(R1.R2)^c$ = {(4,2),(3,3),(2,4)} ---------------- (A)

$R2^c$= {(2,1),(3,2),(4,3)}

$R1^c$ = {(4,1),(3,2),(2,3),(1,4)}

$R2^c. R1^c$ = {(2,4),(3,3),(4,2)} ---------------(B)

Therefore from (A) & (B) we get, **$(R1.R2)^c = R2^c. R1^c$.**

Examples

❖ **Example 5:** Let A be the set of students at your school and B the set of books in the school library. Let R1 and R2 be the relations consisting of all ordered pairs (a, b), where student a is required to read book b in a course, and where student a has read book b, respectively. Describe the ordered pairs in each of these relations.

a) $R1 \cup R2$ b) $R1 \cap R2$ c) $R1 \oplus R2$ d) $R1 - R2$ e) $R2 - R1$

Solution: the set of pairs (a, b) where

a) a is required to read b in a course or has read b.
b) a is required to read b in a course and has read b.
c) a is required to read b in a course or has read b, but not both; equivalently, the set of pairs (a, b) where a is required to read b in a course but has not done so, or has read b although not required to do so in a course.
d) a is required to read b in a course but has not done so.
e) a has read b although not required to do so in a course.

Powers of a Relation

- Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively by **$R^1 = R$ and $R^{n+1} = R^n \circ R$.**
- The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on.

- **Example 6:** Let R = {(1, 1), (2, 1), (3, 2), (4, 3)}. Find the powers R^n, n = 2, 3,4,……

Solution: We have R = {(1, 1), (2, 1), (3, 2), (4, 3)}.

$R^2 = R \circ R = [\{(1, 1), (2, 1), (3, 2), (4, 3)\}]\ [\{(1, 1), (2, 1), (3, 2), (4, 3)\}]$

$R^2 = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$.

$R^3 = R^2 \circ R = [\{(1, 1), (2, 1), (3, 1), (4, 2)\}]\ [\{(1, 1), (2, 1), (3, 2), (4, 3)\}]$

$R^3 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

$R^4 = R^3 \circ R = [\{(1, 1), (2, 1), (3, 1), (4, 1)\}]\ [\{(1, 1), (2, 1), (3, 2), (4, 3)\}]$

$R^4 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$

Powers of a Relation

- **Example 7:** R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.

  Solution: R^1 = {(1,2),(2,3),(2,4), (3,3)}

  R^2 = {(1,3), (1,4), (2,3), (3,3)}

  R^3 = {(1,3), (2,3), (3,3)}

  R^4 = {(1,3), (2,3), (3,3)}

- **Example 8:** Let R be the relation on the set {1, 2, 3, 4, 5} containing the ordered pairs (1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 1), (3, 4), (3, 5), (4, 2), (4, 5), (5, 1), (5, 2), and (5, 4). Find a) R^2. b) R^3. c) R^4. d) R^5.

  Solution: R^1 =

  R^2 =

  R^3 =

  R^4 =

  R^5 =

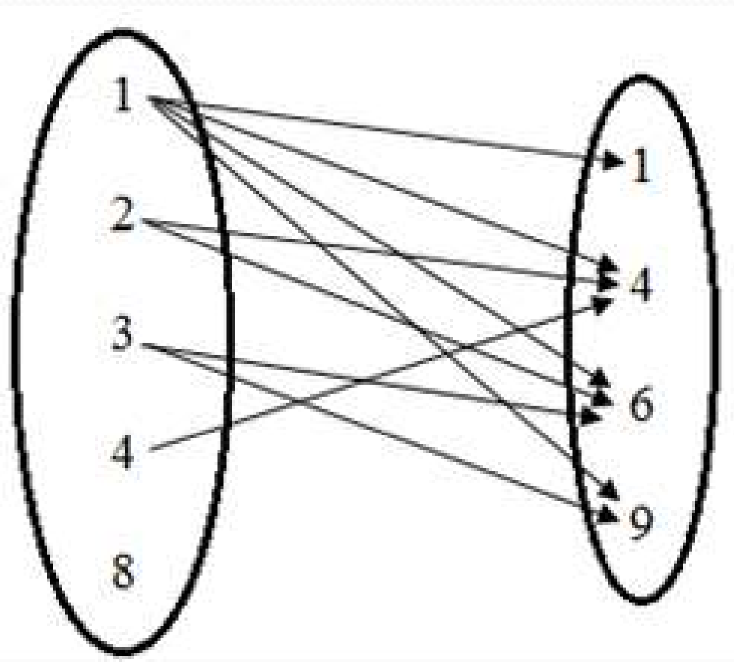
Representation of Relations

❖ Let A =(1,2,3,4,8) & B =(1,4,6,9) R={(x,y) | y is divisible by x}.

Solution: The relation R consists of the ordered pairs:

R={(1,1),(1,4),(1,6),(1,9),(2,4),(2,6),(3,6),(3,9),(4,4)}

❖ 1. Ordered Pair



❖ 2. Tabular Representation

	1	4	6	9
1	*	*	*	*
2		*	*	
3			*	*
4		*		
8				

Representation of Relations

❖ Let A =(1,2,3,4,8) & B =(1,4,6,9) R={(x,y) | y is divisible by x}.
R={(1,1),(1,4),(1,6),(1,9),(2,4),(2,6),(3,6),(3,9),(4,4)}

❖ 3. Matric Representation

✓A relation between finite sets can be represented using a zero–one matrix.

✓The zero–one matrix representing R has 1 as its (i,j) entry when **ai** is related to **bj**, and 0 in this position if **ai** is not related to **bj**.

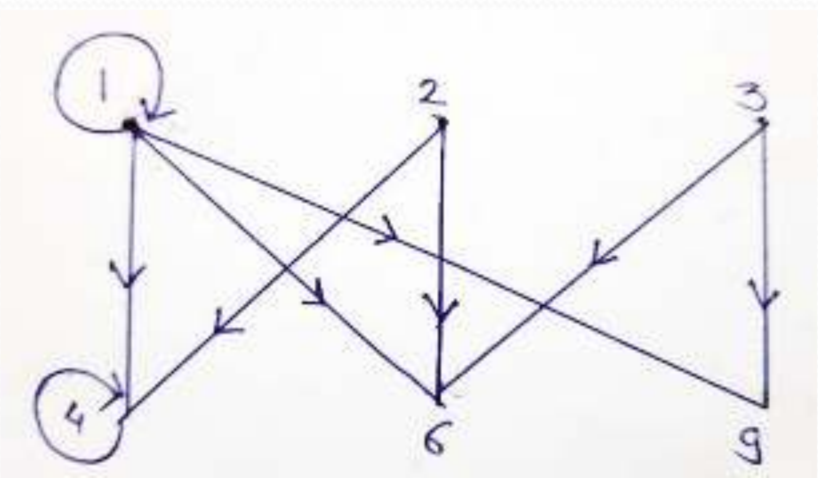
$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

$$M_R = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

❖ 4. Digraph Representation

✓In edge (a,b), vertex a is called the initial vertex and the vertex b is called the terminal vertex.

✓An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called a loop.



Examples

❖ **Example 1:** Let A ={0, 1, 2}, B = {u,v} and R = {(0,u), (0,v), (1,v), (2,u)}. Show all forms of representation of given relation R.

Solution:

R	u	v
0	x	x
1		x
2	x	

or

R	u	v
0	1	1
1	0	1
2	1	0

❖ **Example 2:** Let A = {1,2,3,4}. Define a $R_{\neq}$ b if and only if a ≠ b. What is Inverse Complement of $R_{\neq}$ and Also representation of given relation $R_{\neq}$.

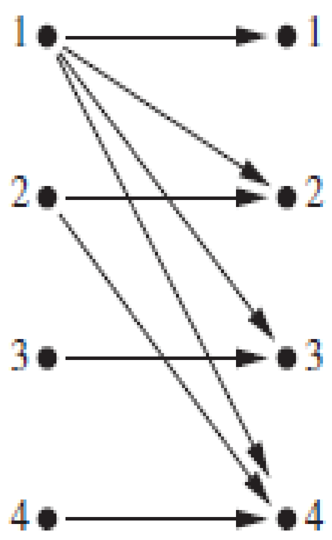
Solution: $R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

R	1	2	3	4
1		x	x	x
2	x		x	x
3	x	x		x
4	x	x	x	

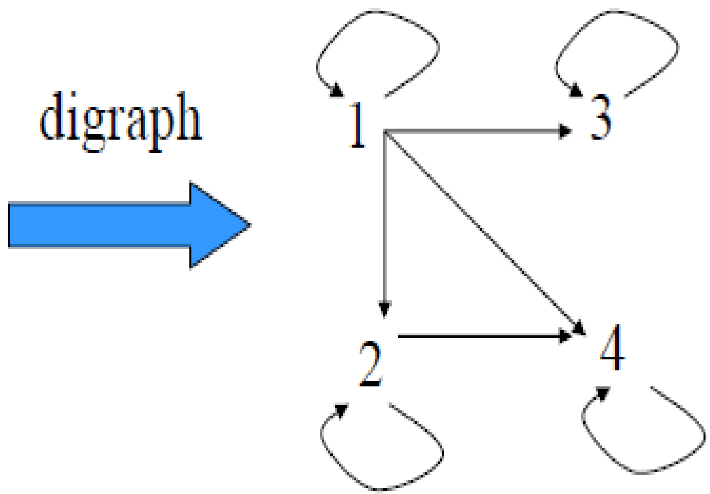
Examples

❖ **Example 3:** Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a divides b}?. Draw all form of representations for R.

Solution: R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}.



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×



Examples

❖ **Example 4:** List the ordered pairs in the relation R from A = {0, 1, 2, 3, 4} to B = {0, 1, 2, 3}, where (a, b) ∈ R if and only if

i. $a = b$.
ii. $a + b = 4$.
iii. $a > b$.
iv. $a \mid b$.

Solution:

i. {(0,0), (1,1), (2,2), (3,3)}

ii. (1,3),(2,2),(3, 1), (4,0)}

iii. {(1,0), (2,0), (2,1),(3,0), (3,1),(3,2),(4,0), (4,1), (4,2), (4,3)}

iv. $a \mid b$ means that b is a multiple of a (a is not allowed to be 0).

{(1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 0), (3, 3), (4, 0)}.

Examples

- **Example 5:** Consider these relations on the set of integers: Which of these relations contain each of the pairs: (1, 1), (1, 2), (2, 1), (1, −1), and (2, 2)?

$R1 = \{(a, b) \mid a \leq b\}$,

$R2 = \{(a, b) \mid a > b\}$,

$R3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,

$R4 = \{(a, b) \mid a = b\}$,

$R5 = \{(a, b) \mid a = b + 1\}$,

$R6 = \{(a, b) \mid a + b \leq 3\}$.

Solution:

(1, 1) → R1, R3, R4 and R6

(1, 2) → R1 and R6

(2, 1) → R2, R5 and R6

(1, −1) → R2 , R3 and R6

(2, 2) → R1 , R3 and R4

Examples

❖ **Example 6:** How many relations are there on a set with n elements.

- **Theorem:** The number of binary relations on a set A, where | A | = n is:

$$2^{n^2}$$

- **Proof:**
- If | A | = n then the cardinality of the Cartesian product | A x A | = n^2.
- R is a binary relation on A if R $\subseteq$ A x A (that is, R is a subset of A x A).
- The number of subsets of a set with k elements : 2^k
- The number of subsets of A x A is : $2^{|AxA|} = 2^{n^2}$

Examples

- **Example**: Let A = {1,2}
- What is A x A = **{(1,1),(1,2),(2,1),(2,2)}**
- **List of possible relations (subsets of A x A):**
- ∅ …. 1
- {(1,1)} {(1,2)} {(2,1)} {(2,2)} …. 4
- {(1,1), (1,2)} {(1,1),(2,1)} {(1,1),(2,2)} …. 6
  {(1,2),(2,1)} {(1,2),(2,2)} {(2,1),(2,2)}
- {(1,1),(1,2),(2,1)} {(1,1),(1,2),(2,2)} …. 4
  {(1,1),(2,1),(2,2)} {(1,2),(2,1),(2,2)}
- {(1,1),(1,2),(2,1),(2,2)} …. 1

16

- Use formula: $\mathbf{2^4 = 16}$

Examples

❖ **Example 7:** How many different relations are there from a set with m elements to a set with n elements?

Solution:

- ✓ There are mn elements of the set A x B, if A is a set with m elements and B is a set with n elements.
- ✓ A relation from A to B is a subset of A x B.
- ✓ Thus the question asks for the number of subsets of the set A x B, which has mn elements.
- ✓ By the product rule, it is $\mathbf{2^{mn}}$**.**

Examples

❖ **Example 9:** Let A = {a1, a2, 3} and B = {b1, b2, b3, b4, b5}. Which ordered pairs are in the relation R represented by the matrix.

Solution:

R = {(a1,b2), (a2,b1), (a2,b3), (a2,b4), (a3,b1), (a3,b3), (a3,b5)}.

$$M_R = \begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{matrix}$$

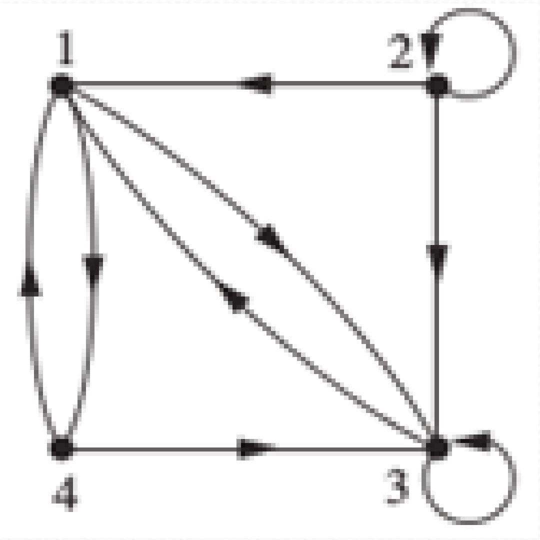
❖ **Example 10:** What are the ordered pairs in the R represented by the directed graph:

Solution:

The ordered pairs (x, y) in the relation are

R = {(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)}.

Each of these pairs corresponds to an edge of the directed graph, with (2, 2) and (3, 3) corresponding to loops.



Representing Relations using Zero–One Matrices

- Let A = [aij] and B = [bij] be m × n zero–one matrices.
- The **Join** of A and B is the zero–one matrix with (i, j)th entry **$a_{ij} \vee b_{ij}$**. The join of A and B is denoted by $A \vee B$.
- The **Meet** of A and B is the zero–one matrix with (i, j)th entry **$a_{ij} \wedge b_{ij}$.** The meet of A and B is denoted by $A \wedge B$.
- **Example:** Let A = {1,2,3} and B = {u,v} and

  R1 = {(1,u), (2,u), (2,v), (3,u)} R2 = {(1,v),(3,u),(3,v)}

$$M_{R1} = \begin{matrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{matrix} \quad M_{R2} = \begin{matrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{matrix} \quad M_{(R1 \vee R2)} = \begin{matrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{matrix} \quad M_{(R1 \wedge R2)} = \begin{matrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{matrix}$$

Matrix Relation for Union and Intersection operation

- The Boolean operations join and meet can be used to find the matrices representing the **union and the intersection of two relations**. Suppose that R_1 and R_2 are relations on a set A represented by the matrices M_{R1} and M_{R2}, respectively.
- The matrix representing the **union** of these relations has a 1 in the positions where either M_{R1} or M_{R2} has a 1.
- The matrix representing the **intersection** of these relations has a 1 in the positions where both M_{R1} and M_{R2} have a 1.
- Thus, the matrices representing the union and intersection of these relations are $\mathbf{M_{R1 \cup R2} = M_{R1} \vee M_{R2}}$ **and** $\mathbf{M_{R1 \cap R2} = M_{R1} \wedge M_{R2}}$

Matrix Relation for Union and Intersection operation

❖ **Example:** Suppose that the relations R1 and R2 on a set A are represented by the matrices. What are the matrices representing R1 ∪ R2 and R1 ∩ R2?

Solution:

$$M_{R1} = \begin{matrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$$

$$M_{R1 \cup R2} = M_{R1} \vee M_{R2} = \begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{matrix}$$

$$M_{R2} = \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix}$$

$$M_{R1 \cap R2} = M_{R1} \wedge M_{R2} == \begin{matrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

Matrix for the Composite of relations

- The matrix for the composite of relations can be found using the Boolean product of the matrices.
- The Boolean product denoted by ʘ, of an m-by-n matrix (a_{ij}) and n-by-p matrix (b_{jk}) of 0s and 1s is an m-by-p matrix (m_{ik}) where

$m_{ik} = 1$, if $a_{ij} = 1$ and $b_{jk} = 1$ for some k=1,2,...,n

0, otherwise.

- Suppose that R is a relation from A to B and S is a relation from B to C. From the definition of the Boolean product, this means that

$$M_{S \circ R} = M_R \odot M_S$$

- The matrix representing the composite of two relations can be used to find the matrix for $M_R{}^n$. In particular

$$M_R n = M_R^{[n]}$$

Matrix for the Composite of relations

- **Example 1:** Let A = {1,2}, {1,2,3} and C = {a,b}.

  R = {(1,2),(1,3),(2,1)} is a relation from A to B.

  S = {(1,a),(3,b),(3,a)} is a relation from B to C.

  R.S = {(1,b),(1,a),(2,a)}

$$M_R = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad M_S = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$$

$$M_R \odot M_S = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix} \odot \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$$

$(0 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 1)$	$(0 \wedge 0) \vee (1 \wedge 0) \vee (1 \wedge 1)$
$(1 \wedge 1) \vee (0 \wedge 0) \vee (0 \wedge 1)$	$(1 \wedge 0) \vee (0 \wedge 0) \vee (0 \wedge 1)$

$0 \vee 0 \vee 1$	$0 \vee 0 \vee 1$
$1 \vee 0 \vee 0$	$0 \vee 0 \vee 0$

$$M_{S \circ R} = M_R \odot M_S = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$$

Matrix for the Composite of relations

❖ **Example 2:** Find the matrix representing the relations S ∘ R, where the matrices representing R and S are

$$M_R = \begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \quad M_S = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{matrix} \quad M_R \odot M_S = \begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \odot \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{matrix}$$

(1 ∧ 0) ∨ (0 ∧ 0) ∨ (1 ∧ 1)	(1 ∧ 1) ∨ (0 ∧ 0) ∨ (1 ∧ 0)	(1 ∧ 0) ∨ (0 ∧ 1) ∨ (1 ∧ 1)
(1 ∧ 0) ∨ (1 ∧ 0) ∨ (0 ∧ 1)	(1∧ 1) ∨ (1 ∧ 0) ∨ (0 ∧ 0)	(1 ∧ 0) ∨ (1 ∧ 1) ∨ (0 ∧ 1)
(0 ∧ 0) ∨ (0 ∧ 0) ∨ (0 ∧ 1)	(0 ∧ 1) ∨ (0 ∧ 0) ∨ (0 ∧ 0)	(0 ∧ 0) ∨ (0 ∧ 1) ∨ (0 ∧ 1)

0 ∨ 0 ∨ 1	1 ∨ 0 ∨ 0	0 ∨ 0 ∨ 1
0 ∨ 0 ∨ 0	1 ∨ 0 ∨ 0	0 ∨ 1 ∨ 0
0 ∨ 0 ∨ 0	0 ∨ 0 ∨ 0	0 ∨ 0 ∨ 0

$$M_{S \circ R} = M_R \odot M_S = \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{matrix}$$

Matrix for the Composite of relations

- **Example:** Find the matrix representing the relation R^2, where the matrix representing R is

$$M_R = \begin{matrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix}$$

Solution: $M_R2 = M_R^{[2]} = \begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{matrix}$

- **Example:** Let R be the relation represented by the matrix. Find the matrices that represent a) R^2. b) R^3. c) R^4.

$$M_R = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$$

Solution: We compute the Boolean powers of M_R; thus

$$M_R2 = M_R^{[2]} = M_R \odot M_R$$

$$M_R3 = M_R^{[3]} = M_R^{[2]} \odot M_R$$

$$M_R4 = M_R^{[4]} = M_R^{[3]} \odot M_R$$

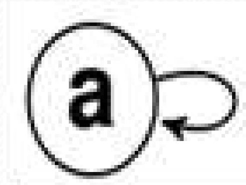
$$R^2 = \begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix} \qquad R^3 = \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \qquad R^4 = \begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

Properties of a Relation

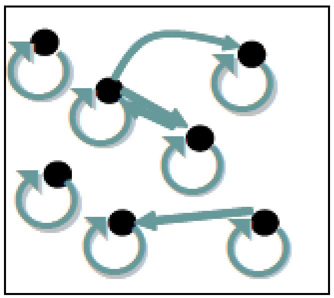
1. **Reflexive Relation:**
2. **Irreflexive Relation:**
3. **Symmetric Relation:**
4. **Anti-Symmetric Relation:**
5. **Asymmetric Relation:**
6. **Transitive Relation:**
7. **Equivalence Relation:**

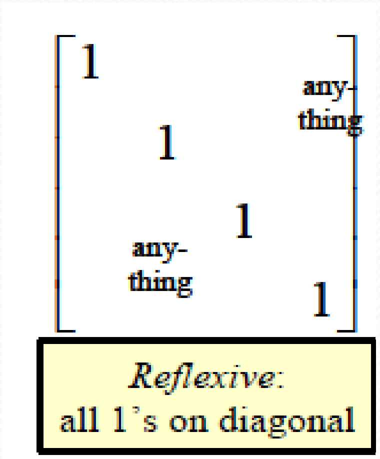
Reflexive Relation

- **Reflexive Relation:** Relation R on a set A is called **reflexive** if $(a,a) \in R$ for every element $a \in A$.



- A relation is **reflexive** if, we observe that for all values a: **aRa.**
- A Relation R is reflexive if all the elements on the Main diagonal of Matrix representation M_R are equal to 1, and the elements off the main diagonal can be either 0 or 1.





- Every Node has a self-loop.
- A relation R is said to be **not reflexive** if there exist at least one element $a \in A$ such that $(a,a) \notin R$. i.e. $a \not R a$

Reflexive Relation

❖ **Example:** Relation R_{div} ={(a,b) if a | b} on A = {1,2,3,4}. Is R_{div} reflexive?

Solution: R_{div} = {(1,1), (1,2), (1,3), (1,4),(2,2), (2,4), (3,3), (4,4)}

R_{div} reflexive because (1,1), (2,2), (3,3), and (4,4) ∈ A.

$$MR_{div} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

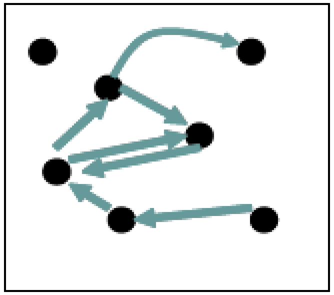
❖ **Example:** Relation R_{fun} on A = {1,2,3,4} defined as:

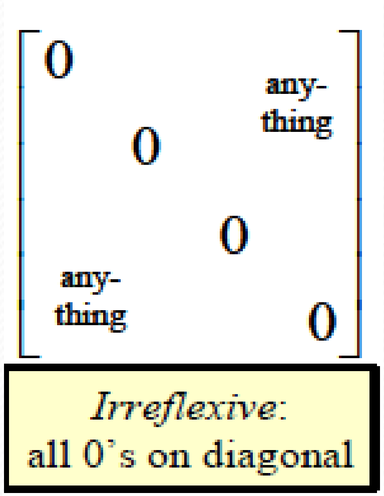
R_{fun} = {(1,2),(2,2),(3,3)}.

Solution: **No.** R_{fun} is not reflexive relation since (1,1) and (4,4) ∉ R_{fun}.

Irreflexive Relation

- ❖ **Irreflexive Relation:** Relation on a set A is called **irreflexive** if $(a, a) \notin R$ for every element $a \in A$.
- ❖ A relation is **Irreflexive** if, we observe that for all values a: aRa does not hold.
- ❖ A relation R is reflexive if and only if Matrix representation (M_R) has 0 in every position on its main diagonal.
- ❖ No node has a self-loop.
- ❖ $R = \{(a, b), (b, a)\}$ on set $X = \{a, b\}$ is irreflexive.





Irreflexive Relation

- **Example:** Relation $R_{\neq}$ on A = {1,2,3,4}, such that

  $aR_{\neq} b$ if and only if $a \neq b$.

  Solution:

  $R_{\neq}$ = {(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

  $R_{\neq}$ **irreflexive** Because (1,1),(2,2),(3,3) and (4,4) $\notin R_{\neq}$

- **Example:** Relation R_{fun} on A = {1,2,3,4} defined as:

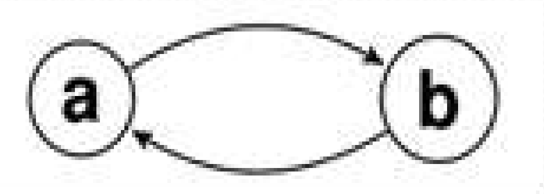
  R_{fun} = {(1,2),(2,2),(3,3)}. Is R_{fun} irreflexive?

  Solution: **No,** Because (2,2) and (3,3) $\in R_{fun}$.

Symmetric Relation

- **Symmetric Relation:** Relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ $\forall (a, b) \in A$.

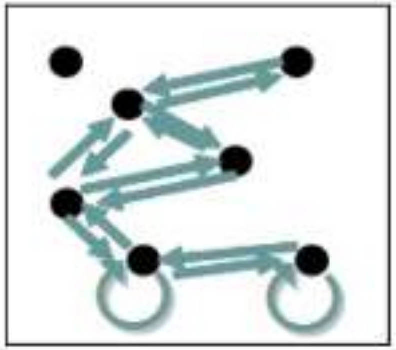
  i. e. $\forall (a, b) \in A \; (a,b) \in R \rightarrow (b,a) \in R$.

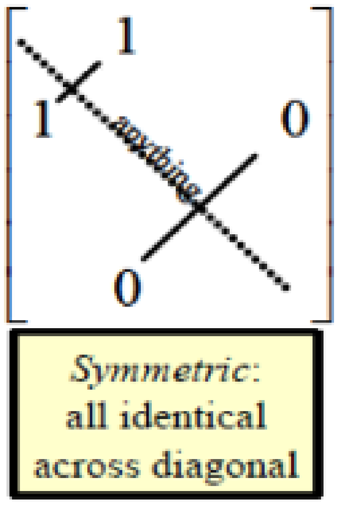


- A relation is **symmetric** if, we observe that for all values of

  a&b: aRb implies bRa

- R is symmetric if and only if $m_{ij} = m_{ji}$, for all pairs of integers i and j. Recalling the definition of the transpose of a matrix, we see that R is symmetric if and only if $M_R=(M_R)^t$, that is, if M_R is a symmetric matrix.





- Every link is Bidirectional.

Symmetric Relation

- **Example:** Let R_{div} ={(a b), if a | b} on A = {1,2,3,4}. Is R_{div} symmetric?

  Solution: R_{div} = {(1,1), (1,2), (1,3), (1,4),(2,2), (2,4), (3,3), (4,4)}

  R_{div} is not symmetric since (1,2) ∈ R but (2,1) ∉ R.

- **Example:** Relation $R_{\neq}$ on A ={1,2,3,4} such that $aR_{\neq}b$ if and only if $a \neq b$.

  Solution: $R_{\neq}$ = {(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

  $R_{\neq}$ symmetric, If (a,b) ∈ $R_{\neq}$ →(b,a) ∈ $R_{\neq}$

0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

- **Example:** Relation R_{fun} on A = {1,2,3,4} defined as:

  R_{fun} = {(1,2),(2,2),(3,3)}. Is R_{fun} symmetric?

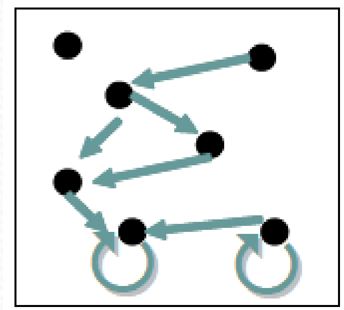
  Solution: It is not symmetric relation since (1,2) ∈ R_{fun} and (2,1) ∉ R_{fun}

Anti-Symmetric Relation

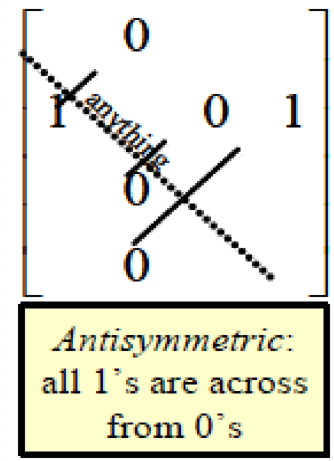
- **Anti-Symmetric Relation:** A relation R on a set A such that for all (a, b) ∈ A, if (a, b) ∈ R and (b, a) ∈ R, then a = b is called **antisymmetric.**

  i. e. **[(a,b) ∈ R and (b,a) ∈ R] → a = b where (a, b) ∈ A.**

- No link is Bidirectional.



- The matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ then $m_{ji} = 0$ for $i \neq j$.
- In other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.



- A relation R on a set A such that for all (a, b) ∈ A, if (a, b) ∈ R and (b, a) ∈ R, then a ≠ b is called **not antisymmetric.**

Anti-Symmetric Relation

- **Imp Notes:**
- A relation is **symmetric** if and only if a R b implies that b R a.
- A relation is **antisymmetric** if and only if there are no pairs of distinct elements a and b with a related to b and b related to a. That is, the only way to have a related to b and b related to a is for a and b to be the same element.
- The terms symmetric and **antisymmetric** are not opposites, because a relation can have both of these properties or may lack both of them. A relation cannot be both symmetric & antisymmetric if it contains some pair of the form (a,b), where a = b.

Anti-Symmetric Relation

- **Example:** Let A = {1,2,3,4} and R_{fun}= {(1,2),(2,2),(3,3)}.

  Is R_{fun} Antisymmetric?

  Solution: Yes R_{fun} antisymmetric since there are no cases of (a, b) and (b, a) in R_{fun}.

$$MR_{fun} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

- **Example:** Let A = {1,2,3,4} and

  R = {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)} Is R Antisymmetric?

  Solution: It is not antisymmetric since it includes both (2,3) and (3,2), but $2 \neq 3$.

Asymmetric Relation

- **Asymmetric Relation**: Relation R on a set A is called **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R \ \forall\ a,b \in A$.

- **Example:** If A = {1, 2, 3} then

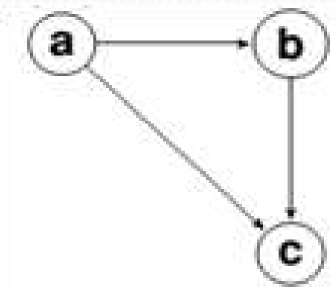
  $R_1 = \{(1,2), (2,3), (3,1)\}$ and $R_2 = \{(1,2), (2,3), (3,2)\}$

  Is R_1 and R_2 Asymmetric?

Solution: R_1 is asymmetric relation and R_2 is not asymmetric relation as (2,3) and $(3,2) \in R_2$.

Transitive Relation

❖ **Transitive Relation:** Relation R on a set A is called transitive if whenever (a, b) ∈ R and (b, c) ∈ R, then (a, c) ∈ R for all a, b, c ∈A.



❖ i. e. [(a,b) ∈ R and (b,c) ∈ R] → (a,c) ∈ R for all a, b, c ∈ A.

❖ **Example:** Is R_{div} $R_{\neq}$ and R_{fun} transitive?

✓ R_{div} = {(1,1), (1,2), (1,3), (1,4),(2,2), (2,4), (3,3), (4,4)}.

✓ $R_{\neq}$ = {(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

✓ R_{fun} = {(1,2),(2,2),(3,3)}.

Solution:

✓ R_{div} is transitive relation as (1,2), (2,4) → (1,4).

✓ $R_{\neq}$ is not transitive since (1,2) ∈ $R_{\neq}$ and (2,1) ∈ $R_{\neq}$ but (1,1) ∉ $R_{\neq}$.

✓ R_{fun} is transitive relation.

Equivalence Relation

- **Equivalence Relation:** A relation is an **Equivalence Relation** if it is reflexive, symmetric, and transitive.
- Two elements a and b that are related by an equivalence relation are called equivalent.
- The notation **a ∼ b or "a ≡ b"** is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.
- **Example:** The relation R = {(1, 1), (2, 2), (3, 3), (1, 2), (2,1), (2,3), (3,2), (1,3), (3,1)} on set A = {1, 2, 3} is an equivalence relation since it is reflexive, symmetric, and transitive.

Equivalence Relation

❖ **Example:** Let R be the relation on the set of real numbers such that aRb if and only if a−b is an integer. Is R an equivalence relation?

Solution: Because a−a =0 is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive.

Now suppose that aRb. Then a−b is an integer, so b−a is also an integer. Hence, bRa. It follows that R is symmetric.

If aRb and bRc, then a−b and b–c are integers. Therefore, a−c=(−b)+(b−c) is also an integer. Hence, aRc. Thus, R is transitive.

Consequently, **R is an equivalence relation.**

Equivalence Classes

- Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a.
- The equivalence class of a with respect to R is denoted by $[a]_R$. i. e.

$$[a]_R = \{s \in A \mid (a, s) \in R\}.$$

- When only one relation is under consideration, we can delete the subscript R and write [a] for this equivalence class.
- If $b \in [a]R$, then b is called a **representative** of this equivalence class.
- Any element of a class can be used as a representative of this class. That is, there is nothing special about the particular element chosen as the representative of the class.
- Any two equivalence classes are either equal or disjoint.

Equivalence Classes

❖ **THEOREM 1:** Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(i) aRb (ii) [a] = [b] (iii) $[a] \cap [b] \neq \emptyset$

❖ **THEOREM 2:** Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i , $i \in I$, as its equivalence classes.

Partition of a Set

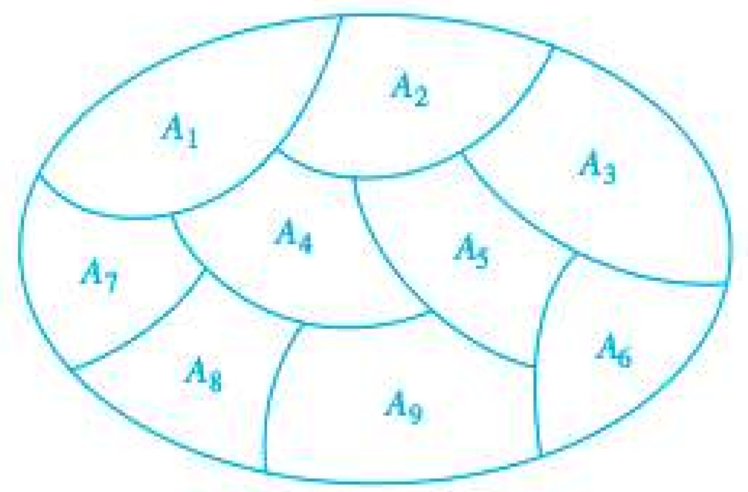
- A partition of a set S is a collection of disjoint nonempty subsets of S that have S as their union.
- In other words, the collection of subsets A_i, $i \in I$ (where I is an index set) forms a partition of S if and only if:

$A_i \neq \emptyset$ for $i \in I$,

$A_i \cap A_j = \emptyset$ when $i \neq j$,

And

$$\bigcup_{i \in I} A_i = S$$



Example on Properties of Relations

- **Example 1:** For each of these relations on the set {1,2,3,4} decide whether it is reflexive, symmetric, antisymmetric, and transitive.

i. {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}

ii. {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

iii. {(2, 4), (4, 2)}

iv. {(1, 2), (2, 3), (3, 4)}

v. {(1, 1), (2, 2), (3, 3), (4, 4)}

vi. {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

Example on Properties of Relations

i. {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}

- ✓ Not reflexive because we do not have (1, 1), and (4, 4).
- ✓ Not symmetric because we we have (2, 4) but not (4, 2). And also we have (3, 4), but not have (4, 3).
- ✓ Not antisymmetric because we have both (2, 3) and (3,2) but $2 \neq 3$.
- ✓ It is Transitive. We can ignore the element 1 since it never appears. If (a, b) is in this relation, then by inspection we see that a must be either 2 or 3. But (2, c) and (3, c) are in the relation for all $c \neq 1$; thus (a, c) has to be in this relation whenever (a, b) and (b, c) are. This proves that the relation is transitive.

Example on Properties of Relations

ii. {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

- ✓ Reflexive because (a, a) is in the relation for all a = 1, 2, 3, 4.
- ✓ Symmetric because that both (1, 2) and (2, 1) are in the relation.
- ✓ Not antisymmetric because both (1, 2) and (2, 1) are in the relation.
- ✓ Transitive because we have (1,2), (2,1) and also (1,1) in the relation.

iii. {(2, 4), (4, 2)}

- ✓ Irreflexive because we do not have (a, a) for all a = 1, 2, 3, 4.
- ✓ Symmetric because for every (a, b), we have (b, a).
- ✓ Not antisymmetric because we have both (2, 4) and (4, 2) in a relation.
- ✓ Not transitive, since although (2,4) and (4,2) are in the relation, (2,2) is not.

Example on Properties of Relation

iv. {(1, 2), (2, 3), (3, 4)}

- ✓ Irreflexive because we do not have (a; a) for all a = 1, 2, 3, 4.
- ✓ Not symmetric because we do not have (2, 1), (3, 2), and (4,3).
- ✓ Antisymmetric because for every (a, b), we do not have (b, a) in relation.
- ✓ Not transitive because we do not have (1,3) for (1, 2) and (2, 3).

v. {(1, 1), (2, 2), (3, 3), (4, 4)}

- ✓ Reflexive because we have (a; a) for every a = 1, 2, 3, 4.
- ✓ Symmetric because we do not have a case where (a, b) and $a \neq b$.
- ✓ Antisymmetric because we do not have a case where (a, b) and $a \neq b$.
- ✓ It is trivially transitive, since the only time the hypothesis $(a, b) \in R \land (b, c) \in R$ is met is when $a = b = c$.

Example on Properties of Relations

vi. {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

- ✓ Irreflexive because we do not have (a; a) for all a = 1, 2, 3, 4.
- ✓ Not symmetric because the relation does not contain (4,1), (3,2), (4,2), and (4,3).
- ✓ Not antisymmetric because we have (1,3) and (3,1).
- ✓ Not transitive because we do not have (2,1) for (2,3) and (3,1).

Example on Properties of Relations

Example 2: Consider the following relations on {1, 2, 3, 4}, decide whether it is reflexive, symmetric, antisymmetric, and transitive.

i. R1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)},

ii. R2 = {(1, 1), (1, 2), (2, 1)},

iii. R3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)},

iv. R4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)},

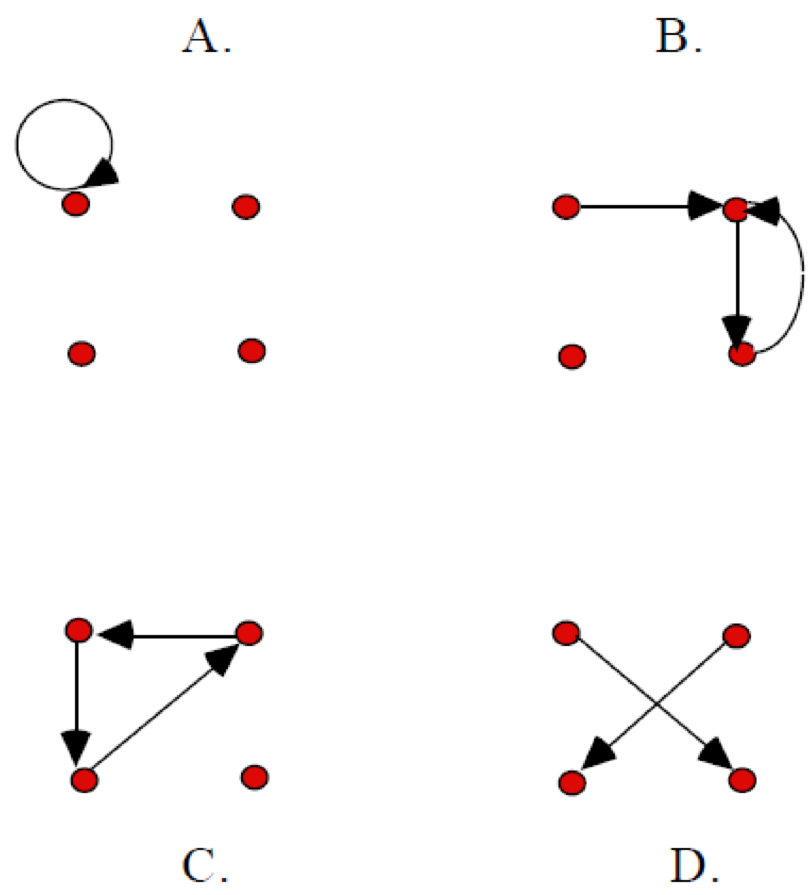
v. R5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)},

vi. R6 = {(3, 4)}.

Relation	Reflexive	Symmetric	Antisymmetric	Asymmetric	Transitive
R1				Y	
R2		Y			
R3	Y	Y			
R4			Y	Y	Y
R5	Y		Y		Y
R6			Y	Y	Y

Example on Properties of Relations

❖ Identify the reflexive, symmetric, antisymmetric, and transitive properties for following digraph representation.



For A:

- ✓ Not Reflexive
- ✓ Symmetric
- ✓ Antisymmetric
- ✓ Transitive

For B:

- ✓ Irreflexive
- ✓ Not Symmetric
- ✓ Not Antisymmetric
- ✓ Not Transitive

For C:

- ✓ Irreflexive
- ✓ Not Symmetric
- ✓ Antisymmetric
- ✓ Not Transitive

For D:

- ✓ Ireflexive
- ✓ Not Symmetric
- ✓ Antisymmetric
- ✓ Transitive

Example on Properties of Relations

❖ **Example 3:** Consider the relation on A= {1, 2, 3, 4, 5, 6}.

R= {(i, j) | i - j = 2}. Is R an Equivalences Relation?

Solution: R={(1,3),(3,1),(2,4),(4,2),(3,5),(5,3),(4,6),(6,4)}

✓ Relation R is not Reflexive as (2,2) is not belong to R.

✓ Relation R is Symmetric and not Transitive Relation.

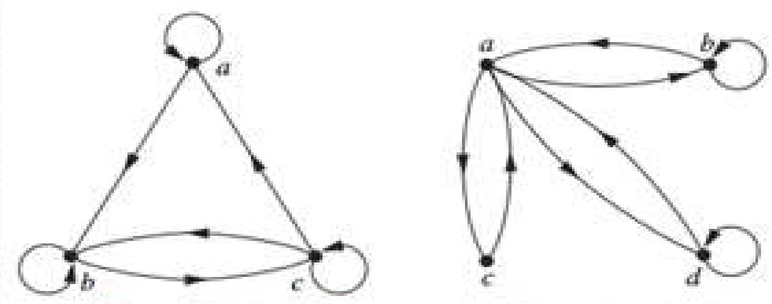
✓ Therefore is not a Equivalences Relation

❖ **Example 4:** Suppose that the relation R on a set is represented by the matrix. Is R reflexive, symmetric, and/or antisymmetric?
Solution: Because all the diagonal elements of this matrix are equal to 1, R is reflexive. Moreover, because M_R is symmetric, it follows that R is symmetric.
It is also easy to see that R is not antisymmetric.

$$M_R = \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{matrix}$$

Example on Properties of Relations

❖ **Example 5:** Determine whether the relations for the directed graphs are reflexive, symmetric, antisymmetric, and/or transitive.



(a) Directed graph of *R* (b) Directed graph of *S*

❖ Relation R is reflexive.

❖ Relation R is neither symmetric nor antisymmetric because there is an edge from a to b but not one from b to a, but there are edges in both directions connecting b and c.

❖ Relation R is not transitive because there is an edge from a to b and an edge from b to c, but no edge from a to c.

❖ Relation S is not reflexive because loops are not present at all the vertices of the directed graph of S.

❖ Relation S It is symmetric and not antisymmetric, because every edge between distinct vertices is accompanied by an edge in the opposite direction.

❖ It is also not hard to see from the directed graph that S is not transitive, because (c, a) and (a, b) belong to S, but (c, b) does not belong to S.

Example on Properties of Relations

❖ **Example 6:** Which of these relations on {0, 1, 2, 3} are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) {(0, 0), (1, 1), (2, 2), (3, 3)}

b) {(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}

c) {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}

d) {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

Example on Properties of Relations

❖ **Example 6: Solution:**

a) This is an equivalence relation; it is easily seen to have all three properties. The equivalence classes all have just one element.

b) This relation is not reflexive since the pair (1, 1) is missing. It is also not transitive, since the pairs (0, 2) and (2, 3) are there, but not (0, 3).

c) This is an equivalence relation. The elements 1 and 2 are in the same equivalence class; 0 and 3 are each in their own equivalence class.

d) This relation is reflexive and symmetric, but it is not transitive. The pairs (1, 3) and (3, 2) are present, but not (1, 2).

e) This relation would be an equivalence relation were the pair (2, 1) present. As it is, its absence makes the relation neither symmetric nor transitive

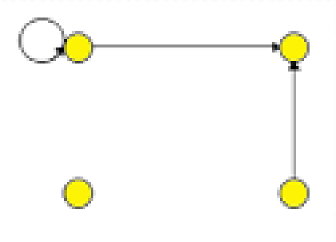
Closures of Relations

- The **Closure of a relation R** with respect to property P is the relation obtained by adding the **minimum number of ordered pairs** to R to obtain property P.
- **In terms of the digraph representation of R**
  - ✓ **To find the reflexive closure - add loops.**
  - ✓ **To find the symmetric closure - add arcs in the opposite direction.**
  - ✓ **To find the transitive closure - if there is a path from a to b, add an arc from a to b.**
- **Note:** Reflexive and symmetric closures are easy. Transitive closures can be very complicated.

Reflexive Closure

- Consider the relation R = {(1, 1), (1, 2), (2, 1), (3, 2)} on the set A = {1, 2, 3} is not reflexive.
- How can we produce a reflexive relation containing R that is as small as possible?
- This can be done by adding (2, 2) and (3, 3) to R, because these are the only pairs of the form (a, a) that are not in R.
- The new relation will be R = {(1, 1), (1, 2), (2, 1), (3, 2), (2, 2), (3, 3)} and is called as reflexive closure of R.
- As this example illustrates, given a relation R on a set A, the reflexive closure of R can be formed by adding to R all pairs of the form (a, a) with a ∈ A, not already in R.

Reflexive Closure

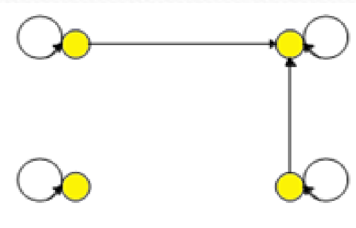


- Reflexive closure of R equals R ∪ Δ,

  where **Δ= {(a, a) | a ∈ A} is the diagonal relation on A.**

- Add loops to all vertices on the digraph representation of R.

- Put 1's on the diagonal of the connection matrix of R.



- **Example:** Let R be the relation on the set A = {0, 1, 2, 3} R= {(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0)}. Find reflexive closure of R.

  Solution: The reflexive closure R = R ∪ Δ, where Δ= {(a, a) | a ∈ A}.

  R = {(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0)}

  Therefore Δ = {(0, 0), (1, 1), (2, 2), (3, 3)}.

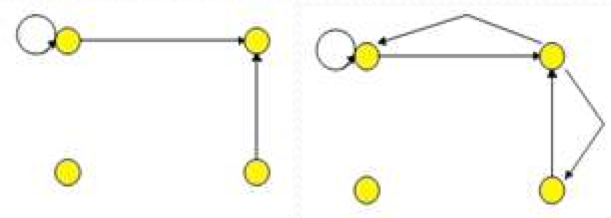
  R = R ∪ Δ = {(0, 0),(0, 1),(1, 1),(1, 2),(2, 0),(2, 2),(3, 0),(3, 3)}.

Symmetric Closure

- Consider the relation R= {(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)} on {1, 2, 3} is not symmetric.
- How can we produce a symmetric relation that is as small as possible and contains R?
- To do this, we need only add (2, 1) and (1, 3), because these are the only pairs of the form (b, a) with (a, b) ∈ R that are not in R.
- This new relation is symmetric and contains R. Furthermore, any symmetric relation that contains R must contain this new relation, because a symmetric relation that contains R must contain (2, 1) and (1, 3). Consequently, this new relation is called the symmetric closure of R.
- Adding these pairs produces a relation that is symmetric, that contains R, and that is contained in any symmetric relation that contains R.

Symmetric Closure

- The **symmetric closure** of a relation can be constructed by taking the union of a relation with its inverse that is, $R \cup R^{-1}$ is the symmetric closure of R, where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.
- Reverse all the arcs in the digraph representation of R.



- Take the transpose MT of the connection matrix M of R.
- **Example:** Let R be the relation on the set A = {0, 1, 2, 3} R= {(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0)}. Find symmetric closure of R.

  Solution: The symmetric closure of $R = R \cup R^{-1}$, where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

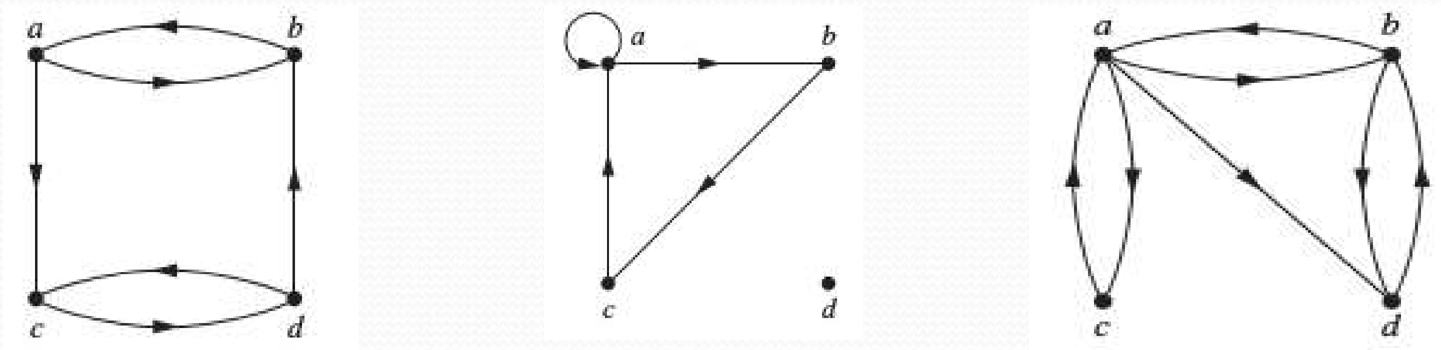
$$R = \{(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0)\}$$

$$R^{-1} = \{(1, 0), (1, 1), (2, 1), (0, 2), (2, 2), (0, 3)\}$$

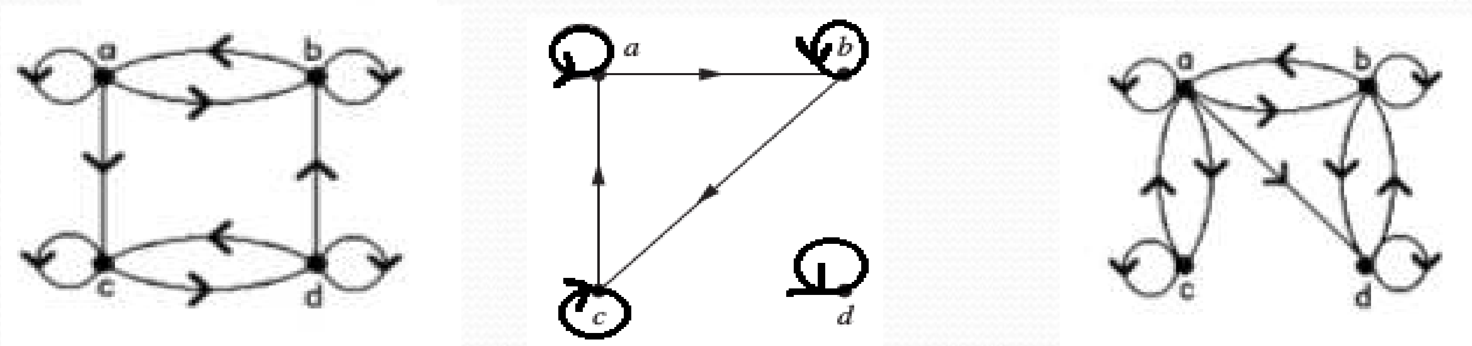
$$\mathbf{R = R \cup R^{-1} = \{ (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (2, 0),(2 ,1), (2, 2), (3, 0)\}.}$$

Example of Reflexive & Symmetric Closure

❖ **Example:** Draw the directed graph of the reflexive closure and Symmetric closure of the relations with the directed graph shown.

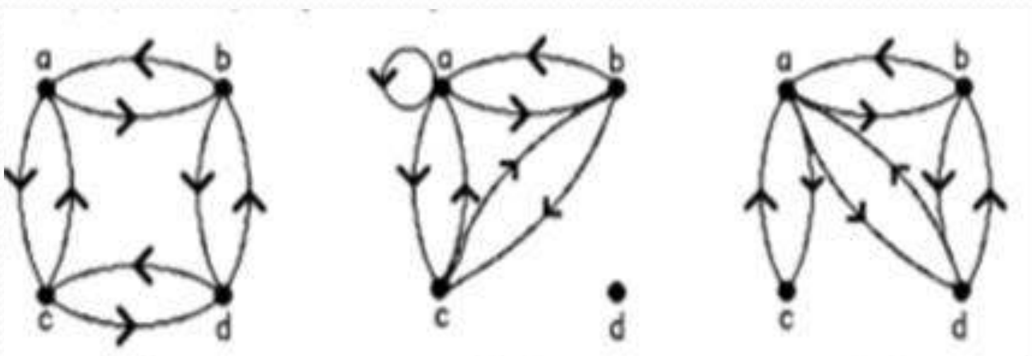


Directed Graph



Reflexive Closure

Add loops to all vertices on the digraph representation of R



Symmetric Closure

Reverse all the arcs in the digraph representation of R

Transitive Closure

- Suppose that a relation R is not transitive.
- How can we produce a transitive relation that contains R such that this new relation is contained within any transitive relation that contains R?
- Can the transitive closure of a relation R be produced by adding all the pairs of the form (a, c), where (a, b) and (b, c) are already in the relation?
- Consider the relation R = {(1, 3), (1, 4), (2, 1), (3, 2)} on the set {1, 2, 3, 4}.
- This relation is not transitive because it does not contain all pairs of the form (a, c) where (a, b) and (b, c) are in R.

Transitive Closure

R = {(1, 3), (1, 4), (2, 1), (3, 2)}

- The pairs of this form not in R are (1, 2), (2, 3), (2, 4), and (3, 1).

R = {(1, 3), (1, 4), (2, 1), (3, 2), (1, 2), (2, 3), (2, 4), (3, 1)}

- Adding these pairs does not produce a transitive relation, because the resulting relation contains (3, 1) and (1, 4) but does not contain (3, 4).
- This shows that constructing the transitive closure of a relation is more complicated than constructing either the reflexive or symmetric closure.
- The transitive closure of a relation can be found by adding new ordered pairs that must be present and then repeating this process until no new ordered pairs are needed.

Transitive Closure

- **Theorem 1**: Let R be a relation on a set A. The connectivity relation R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R. Because R_n consists of the pairs(a, b)such that there is a path of length n from a to b, it follows that $R*$ is the union of all the sets R_n. In other words,

$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

$$= R \cup R^2 \cup R^3 \cup R^4 \cup R^5 \cup \ldots\ldots \cup R_n.$$

- **Theorem 2**: The transitive closure of a relation R equals the connectivity relation $R*$.

- **Theorem 3**: Let MR be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure $R*$ is

$$M_{R*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \cdots \vee M_R^{[n]}$$

Transitive Closure

❖**Example:** A={1,2,3,4}R={(1,2),(1,4),(2,3),(3,4)}.Find its transitive closure

Solution: Let R be a relation on a set A and R* be transitive closure.

$R^* = R \cup R^2 \cup R^3 \cup R^4$

R = {(1,2),(1,4),(2,3),(3,4)}

$R^{2=}$ R.R= [{(1,2),(1,4),(2,3),(3,4)}] .[{(1,2),(1,4),(2,3),(3,4)]

R^2 = {(1,3),(2,4)}

$R^3 = R^2$.R = [{(1,3),(2,4)}].[{(1,2),(1,4),(2,3),(3,4)}].

R^3 = {(1,4)}

$R^4 = R^3$.R= [{(1,4)}].[{(1,2),(1,4),(2,3),(3,4)]

$R^4 = \emptyset$

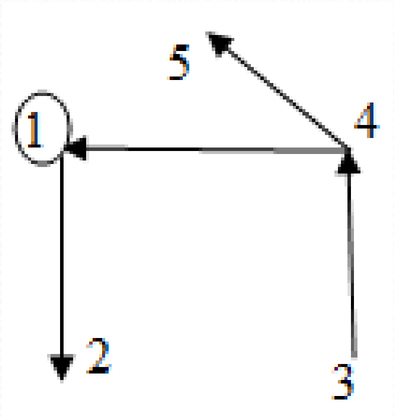
$R^* = R \cup R^2 \cup R^3 \cup R^4$ = {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}.

Transitive Closure

❖ **Example:** If A = {1, 2, 3, 4, 5}, R = {(1,2), (3,4), (4,5), (4,1), (1,1)}. Find its transitive closure.

Solution: Let R be a relation on a set A and R* be transitive closure.

$R^* = R \cup R^2 \cup R^3 \cup R^4 \cup R^5$



R = {(1,2),(3,4),(4,5),(4,1),(1,1)}.

R^2 = R.R = {(3,5) (3,1),(4,2),(4,1),(1,2),(1,1)}

$R^3 = R^2.R$ = {(3,2),(3,1),(4,2),(4,1),(1,1),(1,2)}

$R^4 = R^3.R$ = {(3,1),(3,2),(4,1),(4,2),(1,1),(1,2)}

$R^5 = R^4.R$ = {(3,1),(3,2),(4,1),(4,2),(1,1),(1,2)}

R* = {(3,4), (4,5),(3,1),(3,2),(4,1),(4,2),(1,1),(1,2),(3,5)}

Transitive Closure

❖ **Example:** Find the zero–one matrix of the transitive closure of the relation M_R where

$$M_R = \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix}$$

Solution: From Theorem 3,we have $M_R^* = M_R \vee M_R^{[2]} \vee M_R^{[3]}$

To find $M_R^{[2]}$ we have,

$$M_R^{[2]} = M_R \odot M_R = \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix} \odot \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix}$$

$(1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 1)$	$(1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1)$	$(1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0)$
$(0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 1)$	$(0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1)$	$(0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0)$
$(1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 1)$	$(1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 0)$	$(1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0)$

$1 \vee 0 \vee 1$	$0 \vee 0 \vee 1$	$1 \vee 0 \vee 0$
$0 \vee 0 \vee 0$	$0 \vee 1 \vee 0$	$0 \vee 0 \vee 0$
$1 \vee 0 \vee 0$	$0 \vee 1 \vee 0$	$1 \vee 0 \vee 0$

$$M_R^{[2]} = M_R \odot M_R = \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

Transitive Closure

To find $M_R^{[3]}$ we have,

$$M_R^{[3]} = M_R^{[2]} \odot M_R = \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix} \odot \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix}$$

$(1 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 1)$	$(1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1)$	$(1 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 0)$
$(0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 1)$	$(0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1)$	$(0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0)$
$(1 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 1)$	$(1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1)$	$(1 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 0)$

$1 \vee 0 \vee 1$	$0 \vee 1 \vee 1$	$1 \vee 0 \vee 0$
$0 \vee 0 \vee 0$	$0 \vee 1 \vee 0$	$0 \vee 0 \vee 0$
$1 \vee 0 \vee 1$	$0 \vee 1 \vee 1$	$1 \vee 0 \vee 0$

$$M_R^{[3]} = \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

$$M_R^* = M_R \vee M_R^{[2]} \vee M_R^{[3]} = \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix} \vee \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix} \vee \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

$$M_R^* = \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

Transitive Closures using Warshall's Algorithm

- **Warshall's algorithm** determines whether there is a path between any two nodes in the graph. It does not give the number of the paths between two nodes.
- **Idea:** Compute all paths containing node 1, then all paths containing nodes 1 or 2 or 1 and 2, and so on, until we compute all paths with intermediate nodes selected from the set {1, 2, … n}.
- **Warshall's algorithm** is an efficient method of finding the adjacency matrix of the transitive closure of relation R on a finite set S from the adjacency matrix of R. It uses properties of the digraph D, in particular, walks of various lengths in D.

Warshall's algorithm Steps

- **Step 1:** We have |A| = n. Therefore We require W0,W1,W2,W3,…….Wn

  Warshall sets W0 = Relation Matrix of R = M_R.

- **Step 2:** To find transitive closure of relation R on set A, with |A| = n, compute W_k from W_{k-1} by using following steps:

  a) Copy 1 to all entries in W_k from W_{k-1}, where there is 1 in W_{k-1}.

  b) Find the row numbers R1, R2, R3,….. for which there is 1 in column k in W_{k-1} and column numbers C1, C2, C3,….. for which there is 1 in row k in W_{k-1}.

  c) Mark entries in W_k as 1 for (Ri,Ci). If there are not already 1.

- **Step 3:** Stop the procedure when W_n is obtained and its gives required transitive closure.

Example on Warshall's algorithm

❖ **Example 1:** Use Warshall Algorithm to find the transitive closures, where A = {1,2,3,4,5,6} and R = {(1,3), (2,4), (3,1), (3,5), (4,2), (4,6), (5,3), (6,4)}

Solution:

➢ **Step 1:** |A| = 6.

We have to find W0, W1, W2, W3, W4, W5 and W6 Warshall sets

$$W0 = \begin{matrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{matrix}$$

Example on Warshall's algorithm

- **Step 2:** To find W1 from W0, Consider the first column and first row.

  In R1: 1 is present at C3.

  In C1: 1 is present at R3.

  Thus add new entry in W1 at (R3,C3) =1.

W0 =

0	0	1	0	0	0
0	0	0	1	0	0
1	0	0	0	1	0
0	1	0	0	0	1
0	0	1	0	0	0
0	0	0	1	0	0

W1 =

0	0	1	0	0	0
0	0	0	1	0	0
1	0	1	0	1	0
0	1	0	0	0	1
0	0	1	0	0	0
0	0	0	1	0	0

Example on Warshall's algorithm

- **Step 3:** To find W2 from W1, Consider the second column and second row.

  In R2: 1 is present at C4.

  In C2: 1 is present at R4.

  Thus add new entry in W2 at (R4,C4) =1.

W1 =

0	0	1	0	0	0
0	0	0	1	0	0
1	0	1	0	1	0
0	1	0	0	0	1
0	0	1	0	0	0
0	0	0	1	0	0

W2 =

0	0	1	0	0	0
0	0	0	1	0	0
1	0	1	0	1	0
0	1	0	1	0	1
0	0	1	0	0	0
0	0	0	1	0	0

Example on Warshall's algorithm

- **Step 4:** To find W3 from W2, Consider the third column and third row.

  In R3: 1 is present at C1, C3, C5.

  In C3: 1 is present at R1, R3, R5.

  Thus add new entry in W3 at (R1,C1), (R1,C3), (R1,C5), (R3,C1), (R3,C3), (R3,C5), (R5,C1), (R5,C3), (R5,C5) =1

$$W2 = \begin{matrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{matrix}$$

$$W3 = \begin{matrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{matrix}$$

Example on Warshall's algorithm

- **Step 5:** To find W4 from W3, Consider the fourth column and fourth row.

  In R4: 1 is present at C2, C4, C6.

  In C4: 1 is present at R2, R4, R6.

  Thus add new entry in W4 at (R2,C2), (R2,C4), (R2,C6), (R4,C2), (R4,C4), (R4,C6), (R6,C2), (R6,C4), (R6,C6) =1.

W3 =

1	0	1	0	1	0
0	0	0	1	0	0
1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0
0	0	0	1	0	0

W4 =

1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1

Example on Warshall's algorithm

- **Step 6:** To find W5 from W4, Consider the fifth column and fifth row.

  In R5: 1 is present at C1, C3, C5.

  In C5: 1 is present at R1, R3, R5.

  Thus add new entry in W5 at (R1,C1), (R1,C3), (R1,C5), (R3,C1), (R3,C3), (R3,C5), (R5,C1), (R5,C3), (R5,C5) =1

W4 =

1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1

W5 =

1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1

Example on Warshall's algorithm

- **Step 7:** To find W6 from W5, Consider the six column and six row.

  In R6: 1 is present at C2, C4, C6.

  In C6: 1 is present at R2, R4, R6.

  Thus add new entry in W6 at (R2,C2), (R2,C4), (R2,C6), (R4,C2), (R4,C4), (R4,C6), (R6,C2), (R6,C4), (R6,C6) =1.

W5 =

1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1

W6 =

1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1

Example on Warshall's algorithm

$$W6 = \begin{matrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{matrix}$$

Hence W6 is the transitive closure

R* = {(1,1), (1,3), (1,5), (2,2), (2,4), (2,6), (3,1), (3,3), (3,5), (4,2), (4,4), (4,6), (5,1), (5,3) (5,5), (6,2), (6,4), (6,6)}

Example on Warshall's algorithm

❖ **Example 2:** Use Warshall Algorithm to find the transitive closures of these relations on {1, 2, 3, 4}: R = {(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)}

Solution:

➢ **Step 1:** |A| = 4.

We have to find W0, W1, W2, W3, and W4 Warshall sets

R = {(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)}

$$W0 = \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{matrix}$$

Example on Warshall's algorithm

- **Step 2:** To find W1 from W0, Consider the first column and first row.

  In R1: 1 is present at C2

  In C1: 1 is present at R2, R4.

  Thus add new entry in W1 at (R2,C2), (R4,C2) =1

$$W0 = \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{matrix}$$

$$W1 = \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{matrix}$$

Example on Warshall's algorithm

- **Step 3:** To find W2 from W1, Consider the second column and second row.

  In R2: 1 is present at C1, C2, C3

  In C2: 1 is present at R1, R2, R4.

  Thus add new entry in W2 at (R1,C1), (R1,C2), (R1,C3), (R2,C1) (R2,C2), (R2,C3), (R4,C1), (R4,C2), (R4,C3) =1.

$$W1 = \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{matrix}$$

$$W2 = \begin{matrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

Example on Warshall's algorithm

- **Step 4:** To find W3 from W2, Consider the third column and third row.

  In R3: 1 is present at C4.

  In C3: 1 is present at R1, R2, R4.

  Thus add new entry in W3 at (R1,C4), (R2,C4), (R4,C4) =1.

$$W2 = \begin{matrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

$$W3 = \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}$$

Example on Warshall's algorithm

- **Step 5:** To find W4 from W3, Consider the fourth column and fourth row.

  In R4: 1 is present at C1, C2, C3, C4

  In C4: 1 is present at R1, R2, R3, R4.

  Thus add new entry in W3 at (R1,C1), (R1,C2), (R1,C3), (R1,C4), (R2,C1), (R2,C2), (R2,C3), (R2,C4), (R3,C1), (R3,C2), (R3,C3), (R3,C4), (R4,C1), (R4,C2), (R4,C3), (R4,C4) = 1.

$$W3 = \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}$$

$$W4 = \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}$$

Example on Warshall's algorithm

$$W4 = \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}$$

Hence W4 is the transitive closure

R* = {(1,1), (1,2), (1,3), (1,4),

(2,1), (2,2), (2,3), (2,4),

(3,1), (3,2), (3,3), (3,4),

(4,1), (4,2), (4,3), (4,4)}

Example on Warshall's algorithm

❖ **Example 3:** Use Warshall Algorithm to find the transitive closures of these relations on A = {1, 2, 3, 4}

a) R = {(2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)}

$$\begin{bmatrix}0&0&0&0\\1&0&1&0\\1&0&0&1\\1&0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&0&0\\1&0&1&0\\1&0&0&1\\1&0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&0&0\\1&0&1&0\\1&0&0&1\\1&0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&0&0\\1&0&1&1\\1&0&0&1\\1&0&1&1\end{bmatrix} \quad \begin{bmatrix}0&0&0&0\\1&0&1&1\\1&0&1&1\\1&0&1&1\end{bmatrix}$$

b) R = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}

$$\begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix} \quad \begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix} \quad \begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix} \quad \begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix} \quad \begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

c) R = {(1, 1), (1, 4), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 2)}

$$\begin{bmatrix}1&0&0&1\\1&0&1&0\\1&1&0&1\\0&1&0&0\end{bmatrix} \quad \begin{bmatrix}1&0&0&1\\1&0&1&1\\1&1&0&1\\0&1&0&0\end{bmatrix} \quad \begin{bmatrix}1&0&0&1\\1&0&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix} \quad \begin{bmatrix}1&0&0&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$$

Example on Warshall's algorithm

❖ **Example 4:** Find the transitive closure of the relation R on A={1, 2, 3, 4) defined by R = {(1,2), (1,3), (1,4), (2,1), (2,3), (3,4), (3,2), (4,2),(4,3)}.

❖ **Example 5:** Warshall's algorithm to compute the transitive closure of R∪S for the relations R and S defined on A = {1,2,3,4} described as:

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad M_S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Partial Orderings

- A relation R on a set S is called a **partial ordering or partial order** if it is reflexive, antisymmetric, and transitive.
- A set S together with a partial ordering R is called a partially ordered set, or POSET, and is denoted by (S, R).
- Members of S are called elements of the POSET.
- Partial orderings are used to give an order to sets that may not have a natural one.
- Example: Let S = {1,2,3,4,5,6} and R = {(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (6,1), (6,4), (1,4), (6,5), (3,4), (6,2)}. Then R is partial order on S, and (S,R) is a poset.

Partial Orderings

❖ **Example 1:** Show that the "greater than or equal" relation (≥) is a partial ordering on the set of integers.

Solution:

✓ Because a ≥ a for every integer a, ≥ is reflexive.

✓ If a ≥ b and b ≥ a, then a = b. Hence, ≥ is antisymmetric.

✓ Finally, ≥ is transitive because a ≥ b and b ≥ c imply that a ≥ c.

✓ Thus, ≥ is a partial ordering on the set of integers and **(Z, ≥) is a poset**.

❖ **Example 2:** The divisibility relation | is a partial ordering on the set of positive integers, because it is reflexive, antisymmetric, and transitive, as was shown in Example 3. We see that **(Z+, |) is a poset**. Recall that (Z+ denotes the set of positive integers.)

Partial Orderings

❖ **Example 3:** Is the "divides" relation on the set of positive integers reflexive, symmetric and transitive ?

Solution:

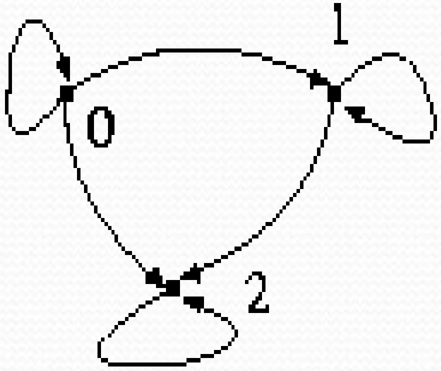
a. Because a | a whenever a is a positive integer, the "divides" relation is **reflexive**. (Note that if we replace the set of positive integers with the set of all integers the relation is not reflexive because by definition 0 does not divide 0.)

b. This relation is not symmetric because 1| 2, but $2 \nmid 1$. It is **antisymmetric**, for if a and b are positive integers with a | b and b | a, then a = b.

c. Suppose that a divides b and b divides c. Then there are positive integers k and l such that b = ak and c = bl. Hence, c = a(kl), so a divides c. It follows that this relation is **transitive**.

Partial Orderings

- ❖ **Example 4:** Show that the inclusion relation ⊆ is a partial ordering on the power set of a set S.
  - ✓ Because A ⊆ A whenever A is a subset of S,⊆ is **reflexive**.
  - ✓ It is **antisymmetric** because A ⊆ B and B ⊆ A imply that A = B.
  - ✓ Finally, ⊆ is **transitive**, because A ⊆ B and B ⊆ C imply that A ⊆ C.
  - ✓ Hence, ⊆ is a partial ordering on P(S), and (P(S),⊆) is a poset.
- ❖ **Example 5:** Let A={0,1,2}and R={(0,0),(0,1),(0,2),(1,1),(1,2),(2,2)}.Show R is a partial order relation.



Solution: The digraph for R on the right implies

Reflexive: Loops on every vertex.

Antisymmetric: No arrows of type (a,b) and (b,a).

Transitive:(0,1), (1,2) also we have (0,2).

Partial Orderings

- **Example 6:** Let R be the relation on the set of people such that xRy if x and y are people and x is older than y. Show that R is not a partial ordering.

  Solution:

  - ✓ Relation R is **antisymmetric** because if a person x is older than a person y, then y is not older than x. That is, if $(x, y) \in R$, then $(y, x) \notin R$.
  - ✓ The relation R is **transitive** because if person x is older than person y and y is older than person z, then x is older than z. That is, if xRy and yRz, then xRz.
  - ✓ Relation R is not **reflexive**, because no person is older than himself or herself. i.e $(x, x) \notin R$ for all people x. It follows that R is not a partial ordering.

Partial Orderings Notation

- A set S together with a partial ordering R is called a partially ordered set, or POSET, and is denoted by (S, R).
- In different posets different symbols such as $\leq$, $\subseteq$, and $|$, are used for a partial ordering.
- Customarily, the notation **$a \preccurlyeq b$** is used to denote that $(a, b) \in R$ in an arbitrary poset (S,R).
- This notation is used because the "less than or equal to" relation on the set of real numbers is the most familiar example of a partial ordering and the symbol $\preccurlyeq$ is similar to the $\leq$ symbol.
- Note that the symbol $\preccurlyeq$ is used to denote the relation in **any poset,** not just the "less than or equals" relation.
- The notation $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$. Also, we say "a is less than b" or "b is greater than a" if $a \prec b$.

Comparable Element

- The elements a and b of a poset $(S,\preceq)$ are called **comparable** if either $a \preceq b$ or $b \preceq a$.
- When a and b are elements of S such that neither $a \preceq b$ nor $b \preceq a$, a and b are called **incomparable**.
- Example: In the poset $(Z^+,|)$, are the integers 3, 9 and 5, 7 comparable?
- The integers 3 and 9 are comparable, because $3 \mid 9$. The integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$.
- The adjective **"partial"** is used to describe partial orderings because pairs of elements may be incomparable. When every two elements in the set are comparable, the relation is called a **total ordering.**

Total Ordered Set & Well-Ordered Set

❖ Total Ordered Set:

- If (S,≼) is a poset and every two elements of S are comparable, S is called a **totally ordered** Or linearly ordered set, and ≼ is called a total order or a linear order. A totally ordered set is also called a **chain.**
- The poset (Z, ≤) is totally ordered, because a ≤ b or b ≤ a whenever a and b are integers.
- The poset (Z+,|) is not totally ordered because it contains elements that are incomparable, such as 5 and 7.
- Therefore Poset (Z, ≼) is a chain and (Z+,|) is not a chain.

❖ Well-Ordered Set:

- (S,≼) is a well-ordered set if it is a poset such that ≼ is a total ordering and every nonempty subset of S has a least element.

Example

- **Example:** Which of these relations on {0, 1, 2, 3} are partial orderings? Determine the properties of a partial ordering that the others lack.

  a. {(0, 0), (1, 1), (2, 2), (3, 3)} Yes- Partial ordering

  b. {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)} Not- Partial ordering

  c. {(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)} Yes- Partial ordering

  d. {(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)} Yes- Partial ordering

  e. {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0),(2, 2), (3, 3)} Not- Partial ordering

- **Example:** Which of these are POSETS?

  a. $(Z, =)$ POSET

  b. $(Z, \neq)$ Not a POSET

  c. $(Z, \geq)$ POSET

  d. $(Z, \nmid)$ Not a POSET

Hasse Diagram

- **A visual representation of a partial ordering.**
- To construct a Hasse diagram for a finite poset (S,⪯), do the following:

1. Start with the directed graph for this relation. Because a partial ordering is reflexive, a loop (a, a) is present at every vertex a. Remove these loops.
2. Next, remove all edges that must be in the partial ordering because of the presence of other edges and transitivity. That is, remove all edges(x,y) for which there is an element z ∈ S such that x≺ z and z ≺ x.
3. Finally, arrange each edge so that its initial vertex is below its terminal vertex.
4. Remove all the arrows on the directed edges, because all edges point "upward" toward their terminal vertex.

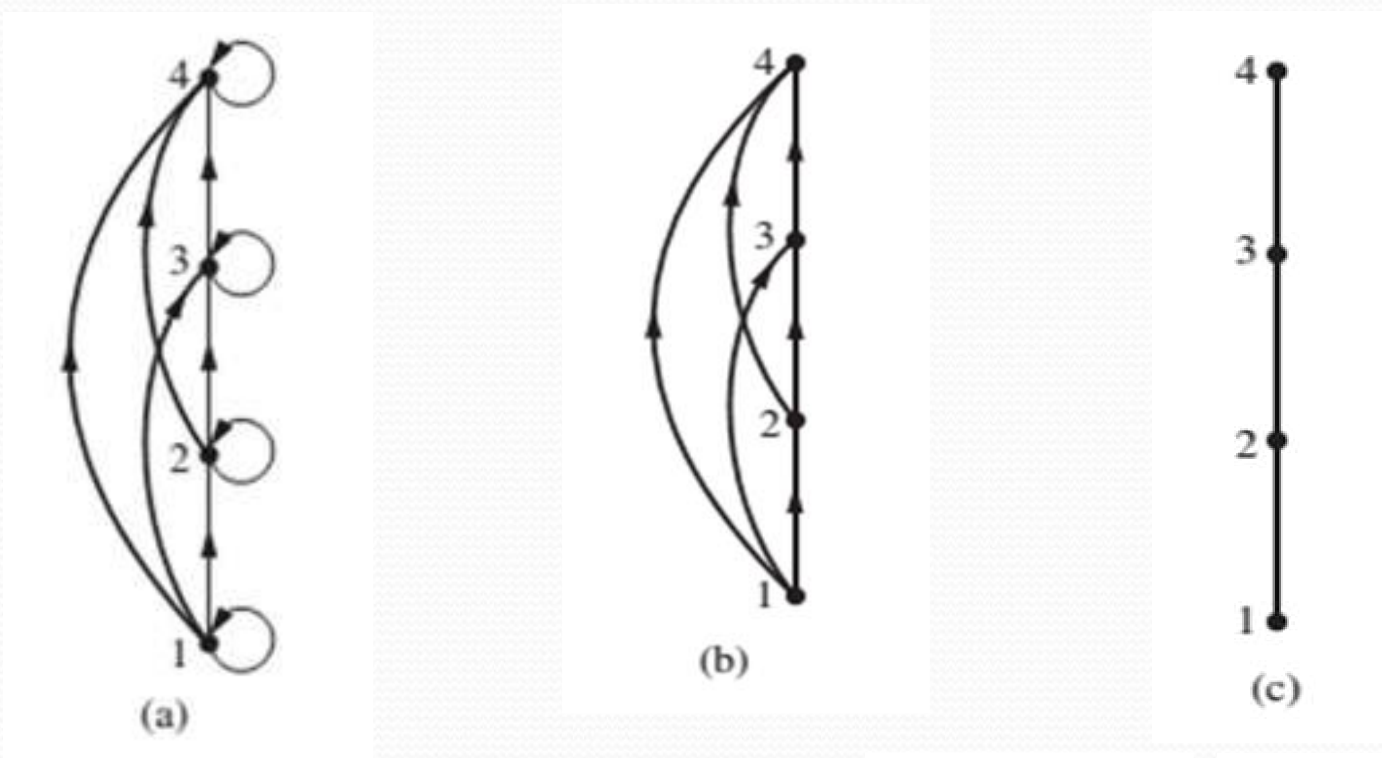
Hasse Diagram-Example

❖ **Example:** Consider the directed graph for the partial ordering {(a, b) | a ≤ b} on the set{1,2,3,4}, shown in Figure (a) and draw Hasse Diagram for it.

Step 1: Remove self loops. i.e. (1,1), (2,2), (3,3) and (4,4) are removed.

Step 2: Remove all edges that must be present because of transitivity. Here edges (1,3), (1,4), and (2,4) are removed.

Step 3: Also remove the arrows, as all arrows point upwards.



Hasse Diagram-Example

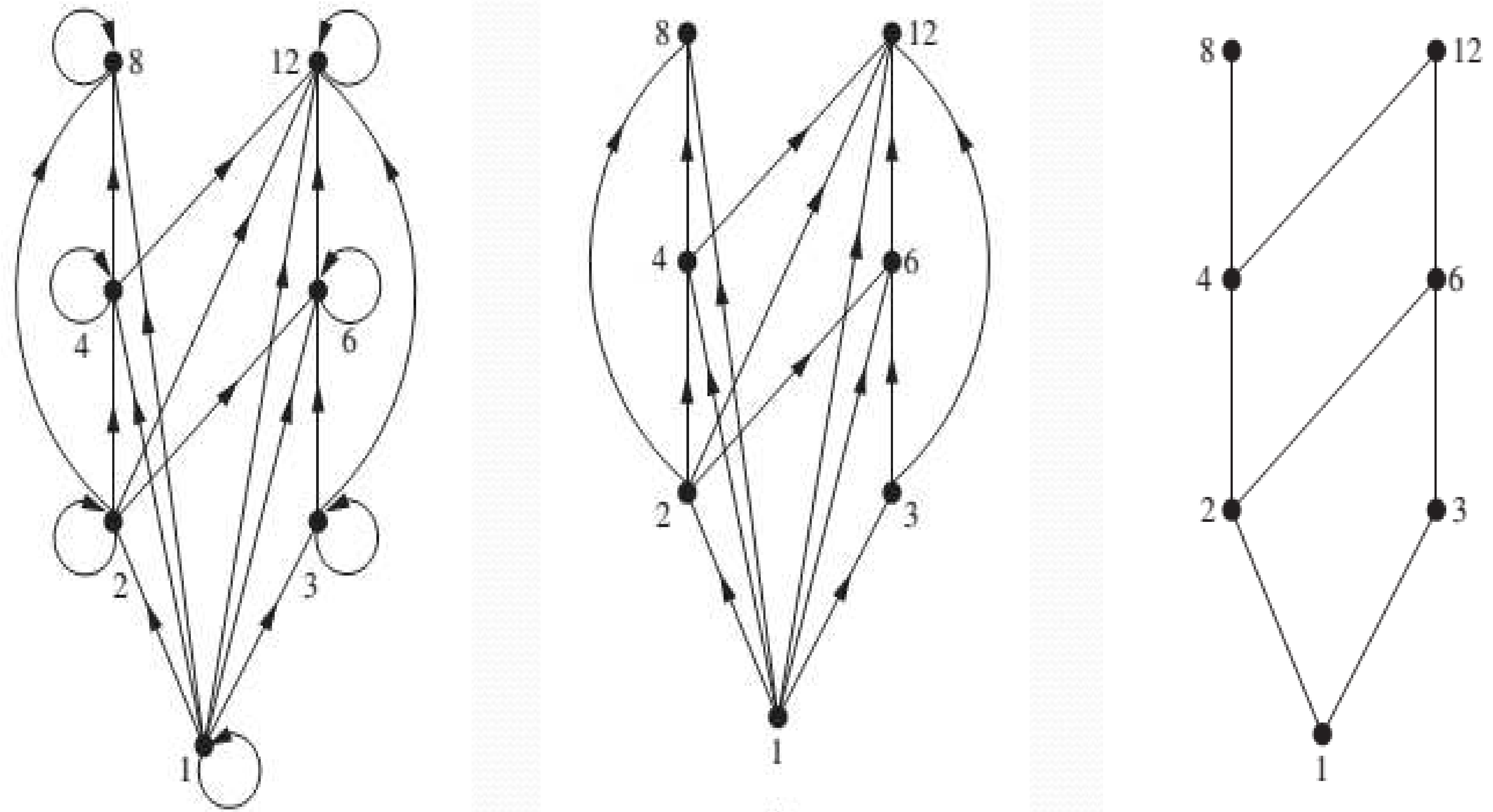
- **Example:** Draw the Hasse diagram representing the partial ordering {(a, b) |a divides b} on {1, 2, 3, 4, 6, 8, 12}.

  Solution: R={(1,1),(1,2),(1,3),(1,4),(1,6),(1,8),(1,12),(2,2),(2,4),(2,6),(2,8),(2,12),(3,3),(3,6),(3,12),(4,4),(4,8),(4,12),(6,6),(6,12),(8,8),((12,12)}.

  - ✓ Begin with the digraph for this partial order, as shown in Fig (a). Remove all loops, as shown in Fig (b).
  - ✓ Then delete all the edges implied by the transitive property. These are (1,4), (1,6), (1,8), (1,12), (2,8), (2,12), and (3,12).
  - ✓ Arrange all edges to point upward, and delete all arrows to obtain the Hasse diagram.
  - ✓ The resulting Hasse diagram is shown in Fig (c).

Hasse Diagram-Example

Solution:



(a) (b) (c)

Hasse Diagram-Example

❖ **Example :** Draw the Hasse diagram for divisibility on the set:

a) {1, 2, 3, 4, 5, 6, 7, 8}
b) {1, 2, 3, 5, 7, 11, 13}
c) {1, 2, 3, 6, 12, 24, 36, 48}
d) {1, 2, 4, 8, 16, 32, 64}
e) {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}

Solution:

We put x above y if y divides x.

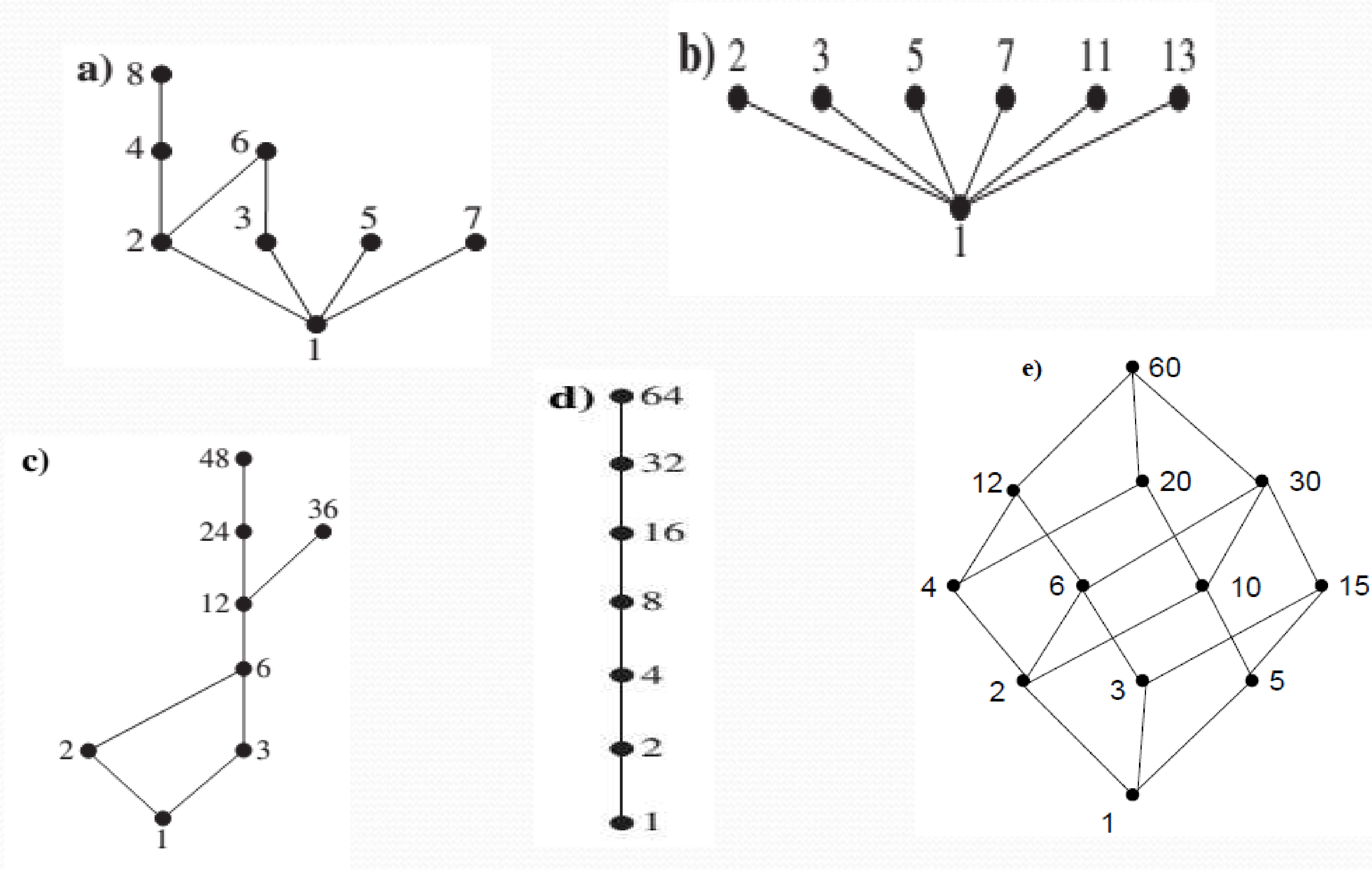
We draw a line between x and y, where y divides x, if there is no number z in our set, other than x or y, such that $y \mid z \wedge z \mid x$.

Note that in part (b) the numbers other than 1 are all (relatively) prime, so the Hasse diagram is short and wide,

Whereas in part (d) the numbers all divide one another, so the Hasse diagram is tall and narrow.

Hasse Diagram-Example

Solution:



Chain and Anti Chain

- A **chain** in a POSET P is a subset C ⊆ P such that any two elements in C are comparable.

- An **antichain** in a POSET P is a subset A ⊆ P Such that no two elements in A are comparable.

Elements of POSETS

❖ **Maximal Elements:**

✓ Let $(A,\preceq)$ be a poset. Then $a \in A$ is maximal in the poset if there is no element $b \in A$ such that $a \prec b$.

❖ **Minimal Elements:**

✓ Let $(A,\preceq)$ be a poset. Then $a \in A$ is minimal in the poset if there is no element $b \in A$ such that $b \prec a$.

❖ **Maximal and Minimal elements** are easy to spot using a Hasse diagram. They are the "top" and "bottom" elements in the diagram. There can be more than one minimal and maximal element in a poset.

Elements of POSETS

❖ **Greatest Element:**

✓ Let $(A,\preceq)$ be a poset. Then $a \in A$ is the greatest element if for every element $b \in A$, $b \preceq a$.

❖ **Least Element:**

✓ Let $(A,\preceq)$ be a poset. Then $a \in A$ is the least element if for every element $b \in A$, $a \preceq b$.

❖ **Upper Bound:**

✓ Let $S \subseteq A$ in the poset $(A,\preceq)$. If there exists an element $u \in A$ such that $s \preceq u$ for all $s \in S$, then u is called an upper bound of S.

Elements of POSETS

❖ **Lower Bound:**

✓ Let $S \subseteq A$ in the poset $(A,\preceq)$. If there exists an element $l \in A$ such that $l \preceq s$ for all $s \in S$, then l is called a lower bound of S.

❖ **Least Upper Bound:**

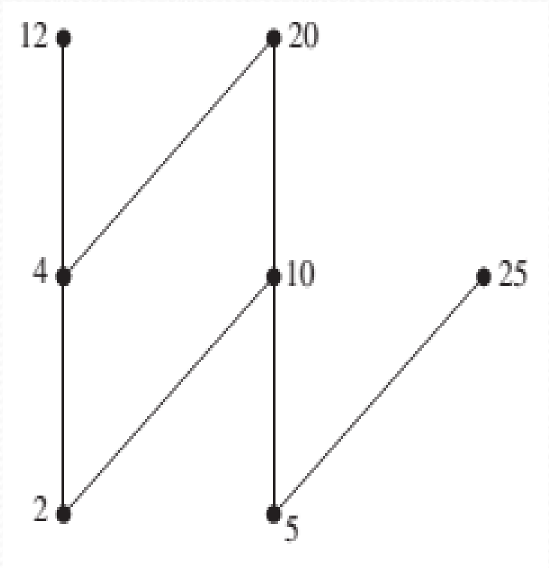
✓ If a is an upper bound of S such that $a \preceq u$ for all upper bound u of S then a is the least upper bound of S, denoted by lub(S).

❖ **Greater Lower Bound:**

✓ If a is a lower bound of S such that $l \preceq a$ for all lower bound l of S then a is the greatest lower bound of S, denoted by glb(S).

Example on Elements of POSETS

❖ **Example 1:** Which elements of the poset ({2, 4, 5, 10, 12, 20, 25},|) are maximal, and which are minimal?

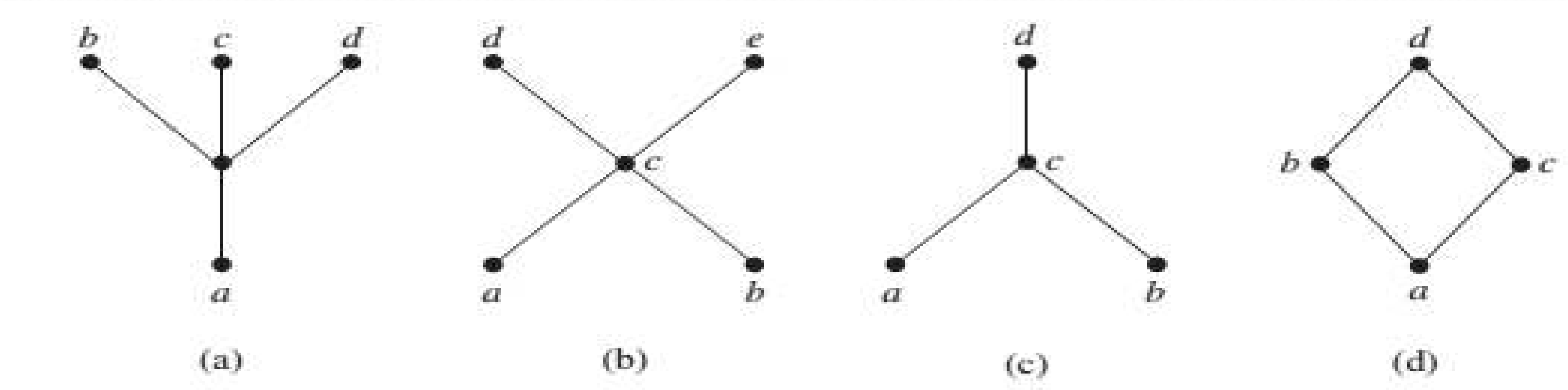


Solution:

- ✓ Maximal elements are 12, 20, and 25, and
- ✓ Minimal elements are 2 and 5.
- ✓ As this example shows, a poset can have more than one maximal element and more than one minimal element.

Example on Elements of POSETS

❖ **Example 2:** Determine whether the posets represented by each of the Hasse diagrams in below figure have a greatest element and a least element.

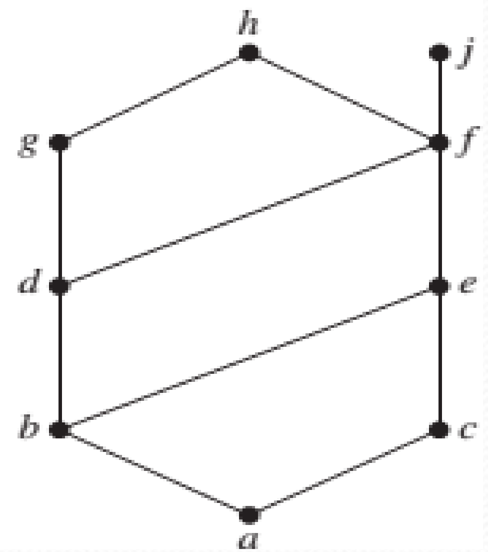


Solution:

- ✓ The least element of the poset with Hasse diagram (a) is a. and has no greatest element.
- ✓ The poset with Hasse diagram (b) has neither a least nor a greatest element.
- ✓ The poset with Hasse diagram (c) has no least element. Its greatest element is d.
- ✓ The poset with Hasse diagram (d) has least element a and greatest element d.

Example on Elements of POSETS

❖ **Example 3:** Find the lower and upper bounds of the subsets {a, b, c}, {j, h}, and {a, c, d, f} in the poset with the Hasse diagram shown in figure below. Also find the glb and the lub of {b, d, g}, if they exist.



Solution:

- ✓ The upper bounds of {a, b, c} are e, f, j, and h, and its only lower bound is a.
- ✓ There are no upper bounds of {j, h}, and its lower bounds are a, b, c, d, e, and f.
- ✓ The upper bounds of {a, c, d, f} are f, h, and j, and its lower bound is a.
- ✓ The upper bounds of {b, d, g} are g and h. Because $g \prec h$, g is the lub. The lower bounds of {b, d, g} are a and b. Because $a \prec b$, b is the glb.

Example on Elements of POSETS

❖ **Example 4:** Answer these questions for the poset ({3, 5, 9, 15, 24, 45}, |).

a) Find the maximal elements. 24, 45

b) Find the minimal elements. 3, 5

c) Is there a greatest element? DNE

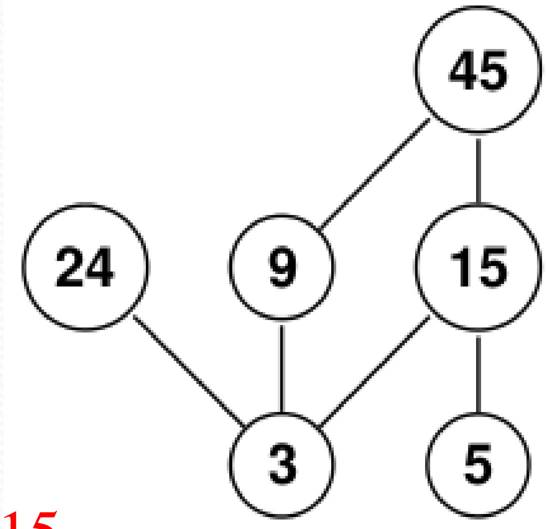
d) Is there a least element? DNE

e) Find all upper bounds of {3, 5}. 15, 45

f) Find the least upper bound of {3, 5}, if it exists. 15

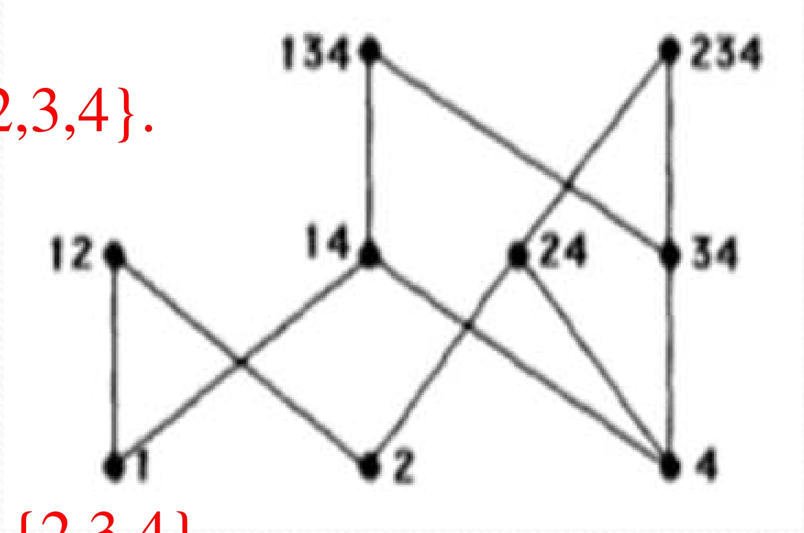
g) Find all lower bounds of {15, 45}. 3, 5, 15

h) Find the greatest lower bound of {15, 45}, if it exists. 15



Example on Elements of POSETS

❖ **Example 5**: Answer the questions for the poset ({{1}, {2}, {4}, {1, 2}, {1, 4}, {2, 4}, {3, 4}, {1, 3, 4}, {2, 3, 4}}, ⊆).



a) Find the maximal elements {1,2}, {1,3,4}, {2,3,4}.

b) Find the minimal elements. { 1} , { 2} { 4}

c) Is there a greatest element? DNE

d) Is there a least element? DNE

e) Find all upper bounds of {{2}, {4}}. {2,4} & {2,3,4}

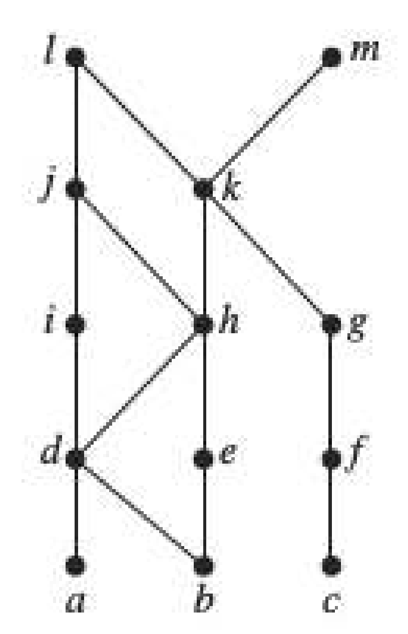
f) Find the least upper bound of {{2}, {4}}, if it exists. {2,4}

g) Find all lower bounds of {{1, 3, 4}, {2, 3, 4}}. {3, 4}, {4}.

h) Find the greatest lower bound of {{1, 3, 4}, {2, 3, 4}},if it exists. {3,4}

Example on Elements of POSETS

❖ **Example 6**:Find the following for given POSET diagram:



a. Find the maximal elements: l, m

b. Find the minimal elements: a, b, c

c. Is there a greatest element?: No

d. Is there a least element?: No

e. Find all upper bounds of {a, b, c } : l, k, m

f. Find the least upper bound of { a, b, c } , if it exists. k

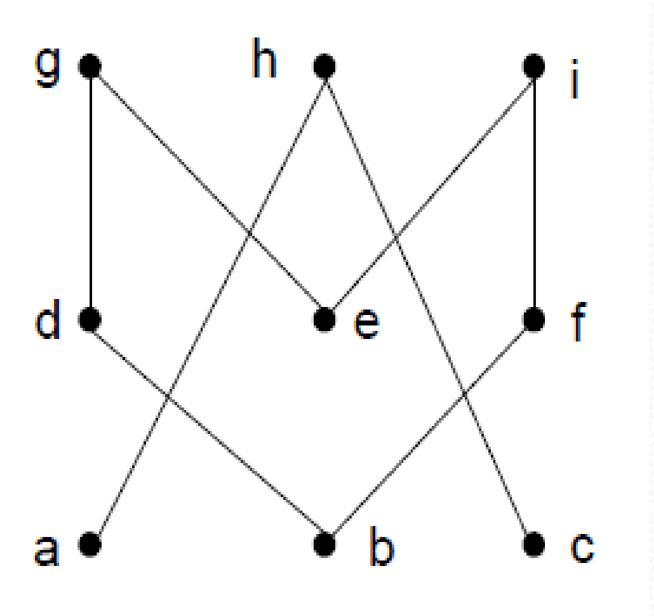
g. Find all lower bounds of {f, g, h}. DNE

h. Find the greatest lower bound of {f, g, h}, if it exists. DNE

Example on Elements of POSETS

❖ **Example 7**: Give lower/upper bounds & glb/lub of the sets:

{d,e,f}, {a,c} and {b,d}.



Solution:

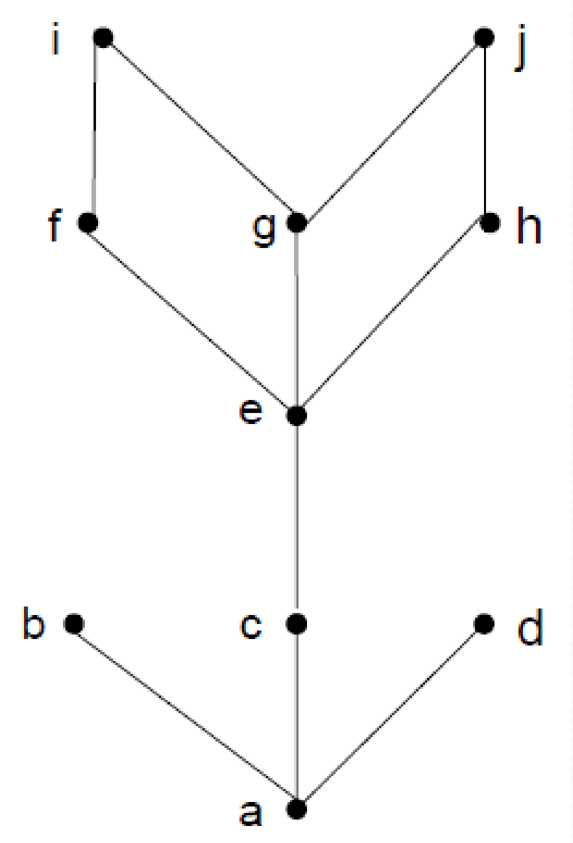
➢ {d,e,f}
  - ✓ Lower bounds: ∅, thus no glb
  - ✓ Upper bounds: ∅, thus no lub

➢ {a,c}
  - ✓ Lower bounds: ∅, thus no glb
  - ✓ Upper bounds: {h}, lub: h

➢ {b,d}
  - ✓ Lower bounds: {b}, glb: b
  - ✓ Upper bounds: {d,g}, lub: d, because $d \prec g$

Example on Elements of POSETS

❖ **Example 8**: Find a) Minimal/Maximal elements? b) Bounds, glb, lub of {c,e}? and c) Bounds, glb, lub of {b,i}?



Solution:

- Minimal/Maximal elements?
  - ✓ Minimal element: a & Maximal elements: b,d,i,j
- Bounds, glb, lub of {c,e}?
  - ✓ Lower bounds: {a,c}, thus glb is c
  - ✓ Upper bounds: {e,f,g,h,i,j}, thus lub is e
- Bounds, glb, lub of {b,i}?
  - ✓ Lower bounds: {a}, thus glb is a
  - ✓ Upper bounds: ∅, thus no lub

Example on Elements of POSETS

❖ **Example 9:** Answer the following questions concerning the poset ({2, 4, 6, 9, 12, 18,27, 36, 48, 60, 72}, |).

a) Find the maximal elements.

b) Find the minimal elements

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of {2, 9} .

f) Find the least upper bound of {2, 9}, if it exists.

g) Find all lower bounds of {60, 72}.

h) Find the greatest lower bound of {60, 72}, if it exists.

Example on Elements of POSETS

❖ **Example 10**:For the Hasse diagram given below; find maximal, minimal, greatest, least, LB, glb, UB, lub for the subsets;

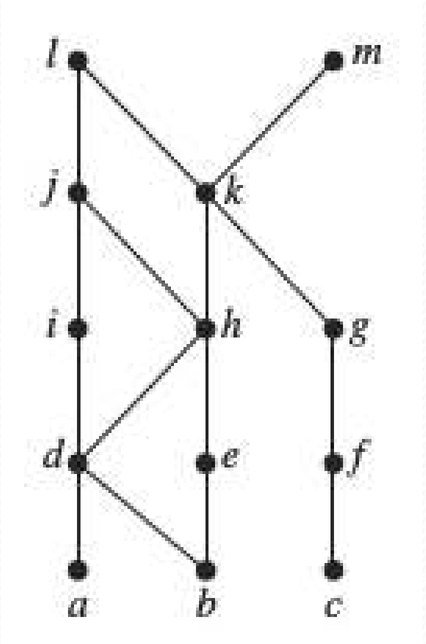
a) {d, k, f}

b) {b, h, f}

c) {d}

d) {a, b, c}

e) {l, m}



Lattices

- A poset in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**.

- A lattice A is called a complete lattice if every subset S of A admits a glb and a lub in A.

- To show that a partial order is not a lattice, it suffices to find a pair that does not have an lub or a glb (i.e., a counter-example).

Lattices

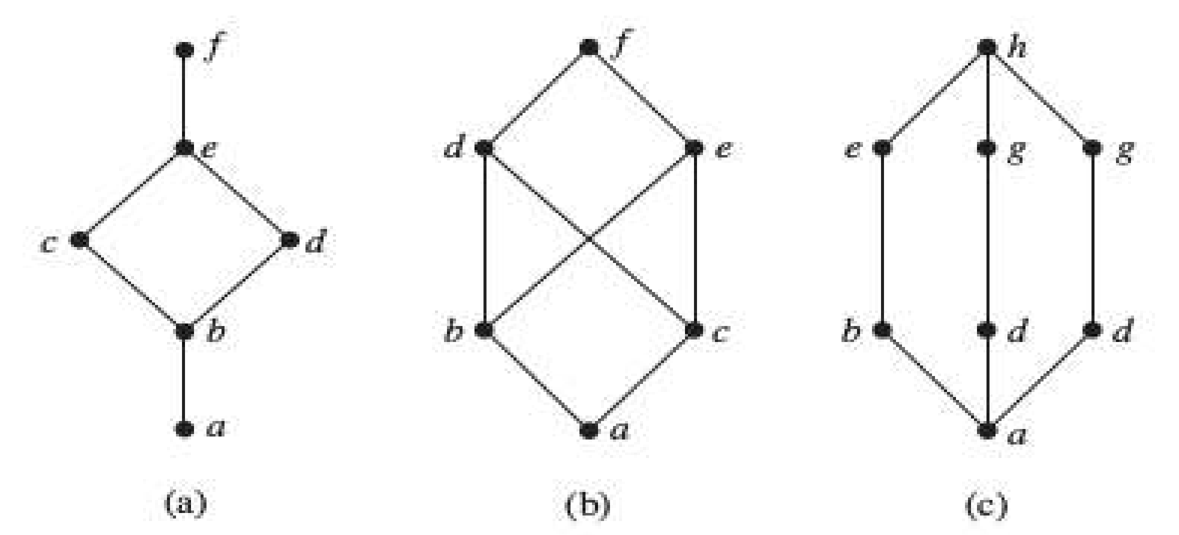
- For a pair not to have an lub/glb, the elements of the pair must first be incomparable.

- You can then view the upper/lower bounds on a pair as a sub-Hasse diagram: If there is no maximum/minimum element in this sub-diagram, then it is not a lattice.

- Lattices have many special properties. Furthermore, lattices are used in many different applications such as models of information flow and play an important role in Boolean algebra.

Lattices

❖ **Example 1:** Determine whether the posets with these Hasse diagrams are lattices.

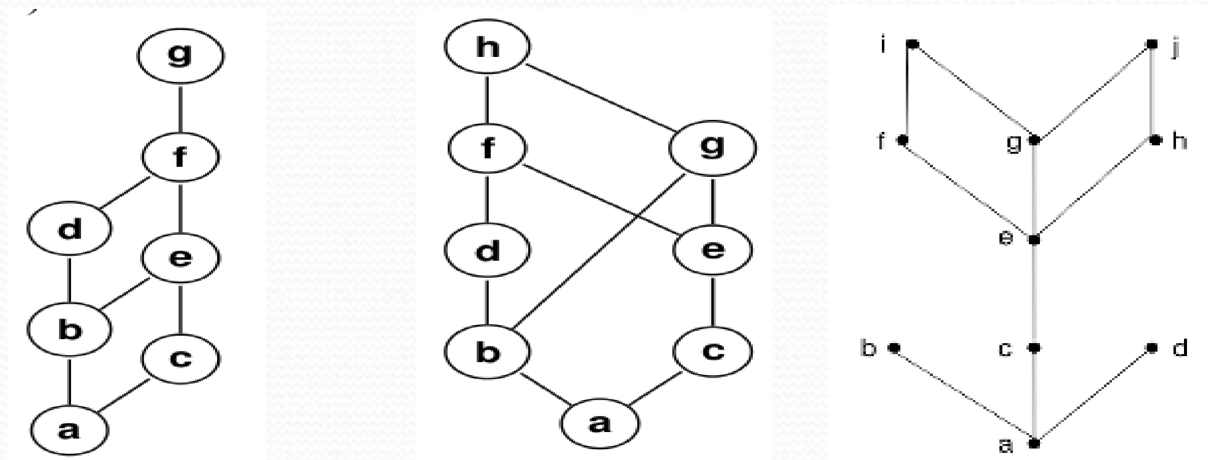


Solution:

- figure a is Lattices.
- figure b is not Lattices because, {b,c} has no lub However, it has a glb={a}.
- figure c is Lattices.

Lattices

❖ **Example 2:** Determine whether the posets with these Hasse diagrams are lattices.

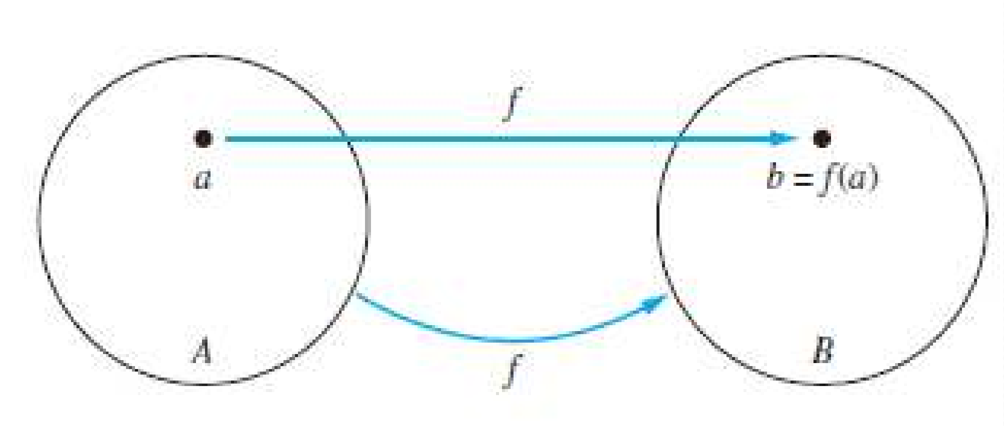


Solution:

- a) Yes. Every two elements will have a least upper bound and greatest lower bound.
- b) No. If we take the elements b and c, then we will have f, g, and h as the upper bound, but none of them will be the least upper bound.
- c) No, because the pair {b,c}does not have a least upper bound.

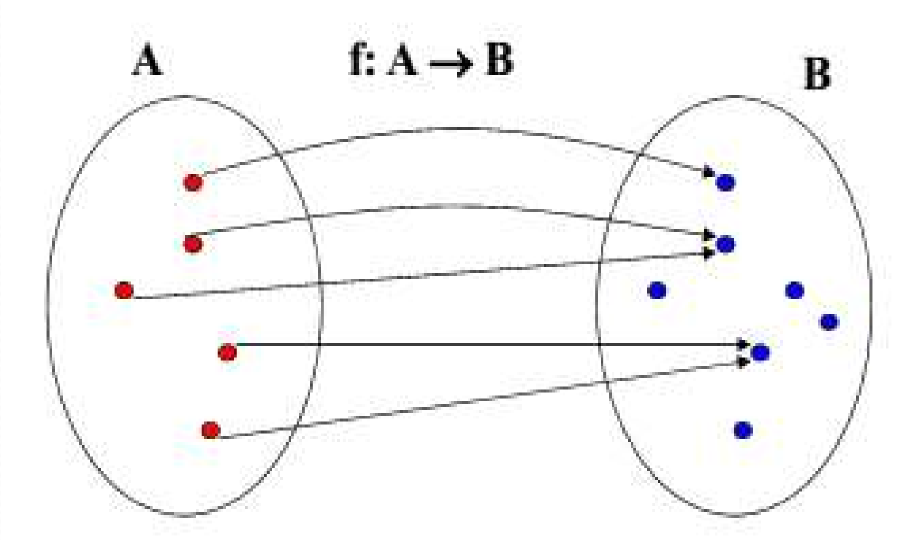
Function-Introduction

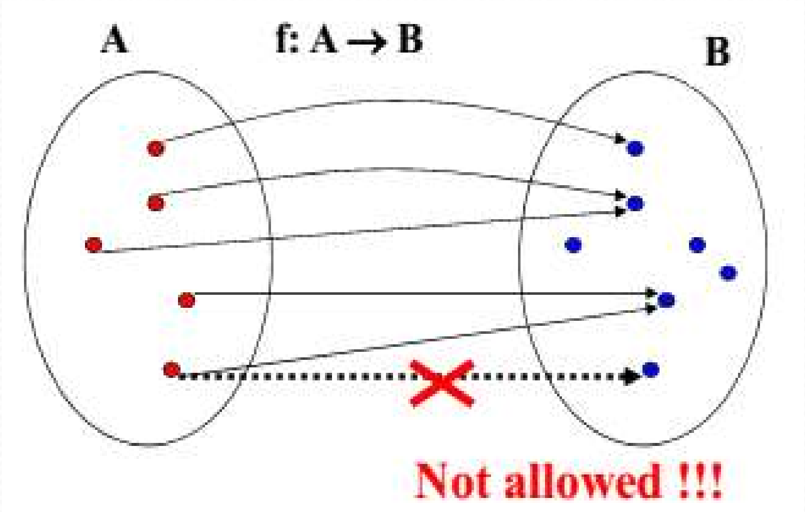
- **Definition ::** Let A and B be nonempty sets. A function f from A to B, denoted ***(f: A →B),*** is an assignment of exactly one element of B to each element of A.
- We write ***f (a) = b*** if b is the unique element of B assigned by the function f to the element a of A.



Function-Introduction

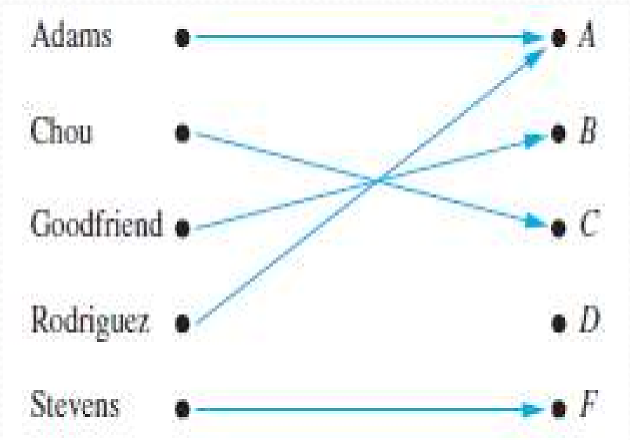
- If ***f*** is a function from A to B, we say that A is the ***domain*** of ***f*** and B is the ***codomain*** of ***f***.
- If f (a) = b, we say that b is the **image** of a and a is a **preimage** of b.
- The range, or image, of *f* is the set of all images of elements of A. Also, if ***f*** is a function from A to B, we say that f maps A to B.





Function-Introduction

❖ **Example 1:** Suppose that each student in a discrete mathematics class is assigned a letter grade from the set {A,B,C,D, F}. And suppose that the grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens.



What are the domain, codomain, and range of the function that assigns grades to students.

Solution: Let G be the function that assigns a grade to a student in our discrete mathematics class. Note that G(Adams) = A, for instance.

The domain of G is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}, and

The codomain is the set {A,B,C,D, F}.

The range of G is the set {A,B,C, F}, because each grade except D is assigned to some student.

Function-Introduction

- **Example 2:** Let R be the relation with ordered pairs (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22). Here each pair consists of a graduate student and this student's age. Specify a function determined by this relation.
  - ✓ **Solution:** If f is a function specified by R, then f (Abdul)= 22, f (Brenda)= 24,f (Carla)= 21, f (Desire)= 22, f (Eddie)= 24, and f (Felicia)= 22.
  - ✓ (Here, f (x) is the age of x, where x is a student.) For the domain, we take the set {Abdul, Brenda, Carla, Desire, Eddie, Felicia}.
  - ✓ We also need to specify a codomain, which needs to contain all possible ages of students.
  - ✓ Because it is highly likely that all students are less than 100 years old, we can take the set of positive integers less than 100 as the codomain.
  - ✓ The range of the function we have specified is the set of different ages of these students, which is the set {21, 22, 24}.

Function-Introduction

- **Example 3:** Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example, f (11010) = 10. Then, the domain of f is the set of all bit strings of length 2 or greater, and both the codomain and range are the set {00, 01, 10, 11}.

- **Example 4:** Let **f : Z → Z** assign the square of an integer to this integer. Then, f (x) = x2, where the domain of f is the set of all integers, the codomain of f is the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, {0, 1, 4, 9, . . . }.

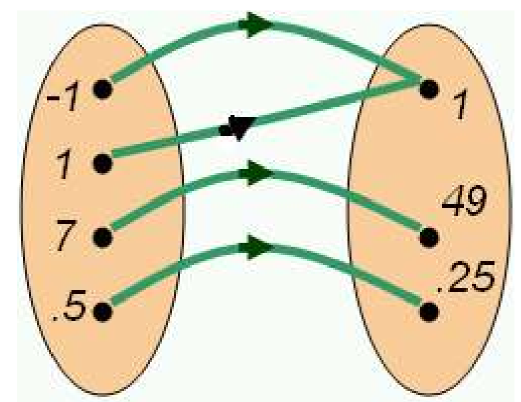
Function-Introduction

❖ **Example 5:** If we write (define) a function as: $f(x)=x^2$ then we say: 'f of x equals x squared' and we have:

$f(-1) = 1$ $f(1) = 1$ $f(2) = 4$ $f(5) = 25$ $f(7) = 49$

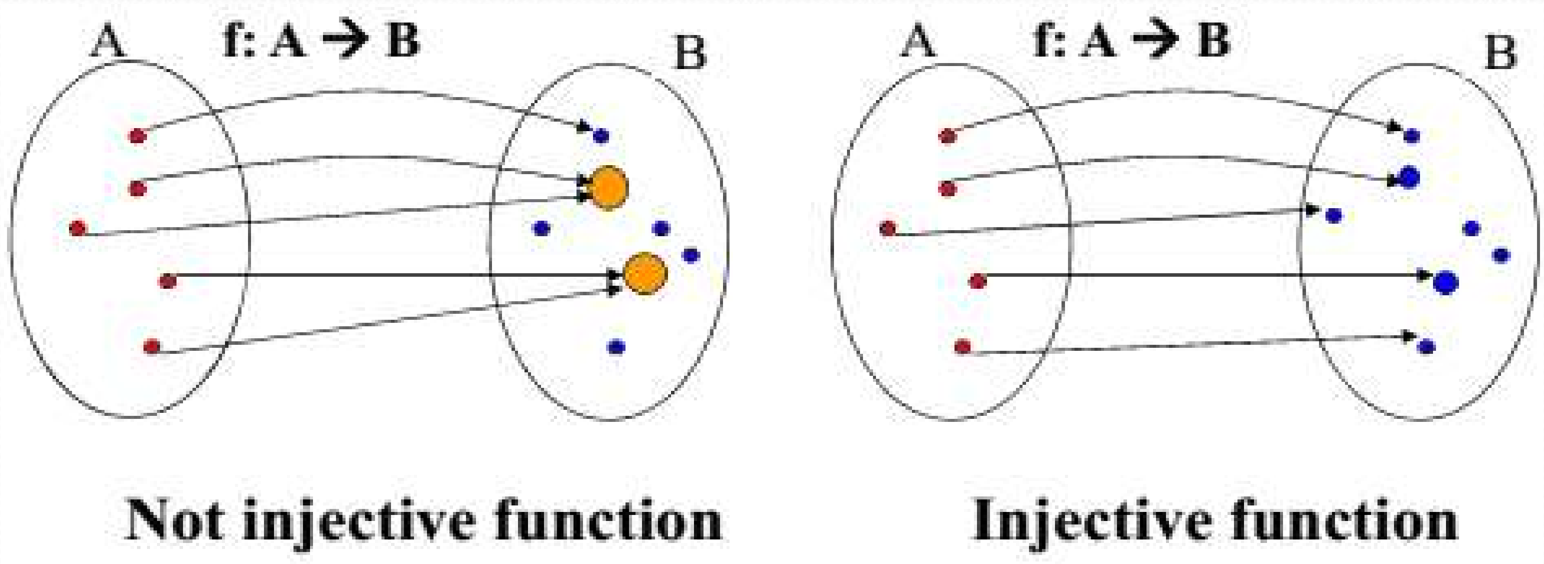
and so on.

This function *f* maps numbers to their squares.



Type of Function

- **Injective / One-to-one function::** A function f is said to be one-to-one, or injective, if and only if f(x) = f(y) implies x = y for all x, y in the domain of f. A function is said to be an injection if it is one-to-one.
- **Alternative:** A function is one-to-one if and only if f(x) ≠ f(y), whenever x ≠ y. This is the contrapositive of the definition.

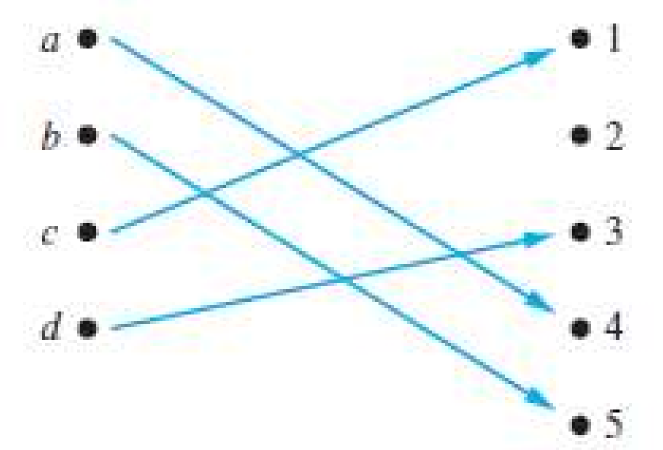


Type of Function

❖ **Injective / One-to-one function::** A function f is said to be one-to-one, or injective, if and only if f(x) = f(y) implies x = y for all x, y in the domain of f. A function is said to be an injection if it is one-to-one.

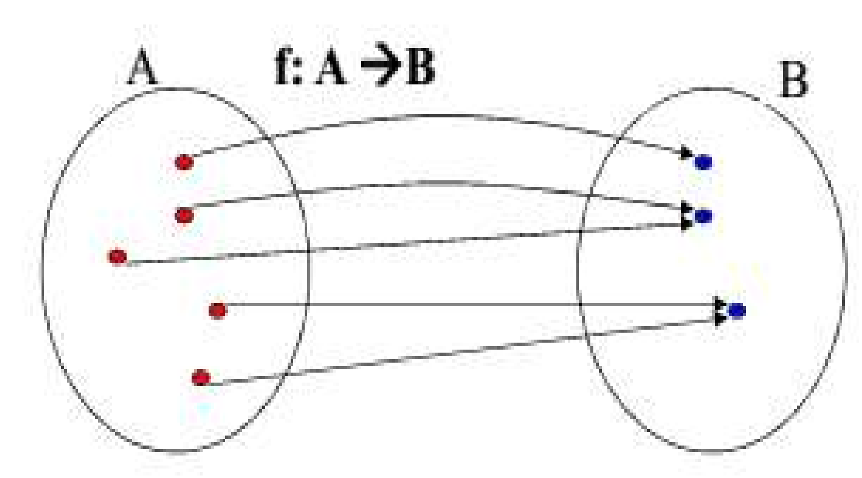
❖ **Example**: Determine whether the function f from {a, b, c, d} to {1, 2, 3, 4, 5} with f(a) = 4, f(b) = 5, f(c) = 1, and f(d) = 3 is one-to-one.

Solution: The function f is one-to-one because ***f*** takes on different values at the four elements of its domain.



Type of Function

- **Surjective / Onto function::** A function f from A to B is called onto, or surjective, if and only if for every element b ∈ B there is an element a ∈ A such that f(a) = b.
- **Alternative:** all co-domain elements are covered.

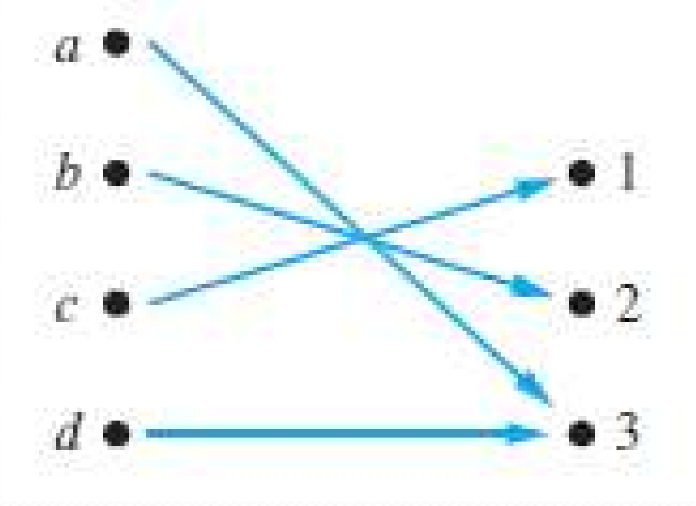


Type of Function

❖ **Surjective / Onto function::** A function f from A to B is called onto, or surjective, if and only if for every element b ∈ B there is an element a ∈ A such that f(a) = b.

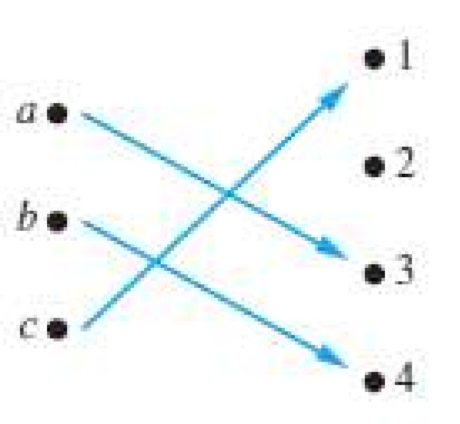
Example: Let f be the function from {a, b, c, d} to {1, 2, 3} defined by f (a) = 3, f (b) = 2, f (c) = 1, and f (d) = 3. Is f an onto function?

Solution: Because all three elements of the codomain are images of elements in the domain, we see that f is onto. Note that if the codomain were {1, 2, 3, 4}, then f would not be onto.

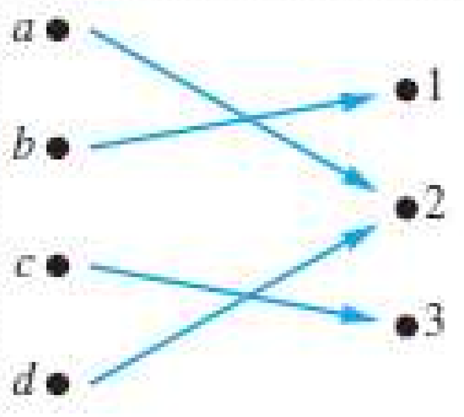


Type of Function

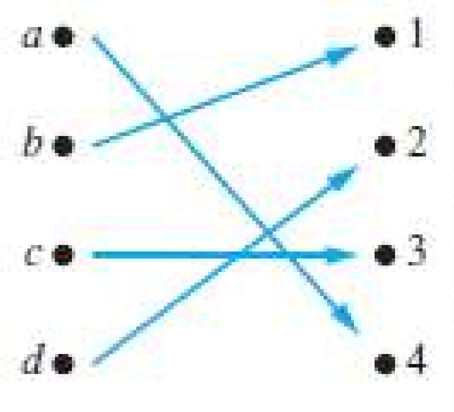
Example: Different Types of Correspondences.



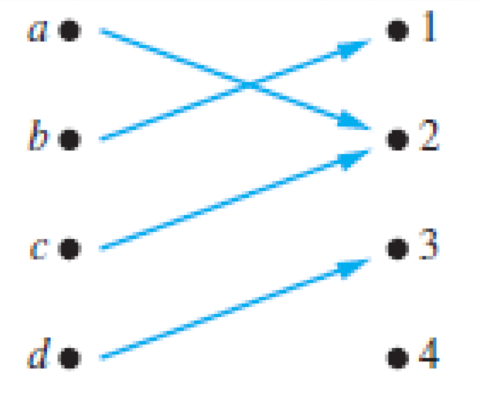
One-to-one, not onto



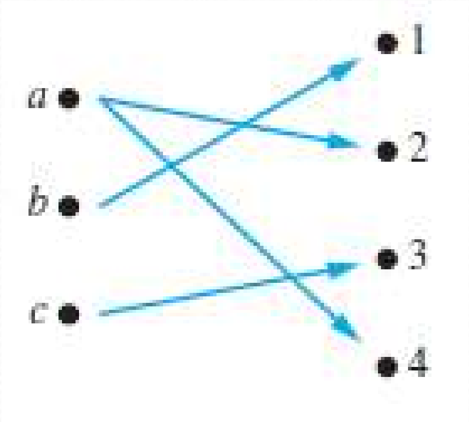
Onto, not one-to-one



One-to-one, and onto



Neither one-to-one, nor onto

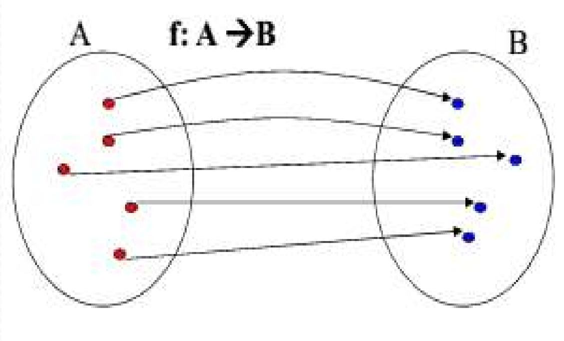


Not a function

Type of Function

❖ **Bijective / One-to-one Correspondent::**A function f is called a bijection if it is both one-to one (injection) and onto (surjection).

❖ **Example:** Let f be the function from {a, b, c, d} to {1, 2, 3, 4}with f (a) = 4, f (b) = 2, f (c) = 1, and f (d) = 3. Is f a bijection?



Solution: The function f is one-to-one and onto. Hence, f is a bijection

❖ **Identity function::** Let A be a set. The identity function on A is the function i_A: A⟶ A where i_A (x) = x.

❖ Example: Let A = {1, 2, 3} Then: $i_A(1) = 1$ $i_A(2) = 2$ and $i_A(3) = 3$.

Type of Function-Summarize

❖ **Suppose that $f : A \to B$.**

❖ ***To show that f is injective***

Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x = y$, then $x = y$.

❖ ***To show that f is not injective***

Find particular elements $x, y \in A$ such that $x = y$ and $f(x) = f(y)$.

❖ ***To show that f is surjective***

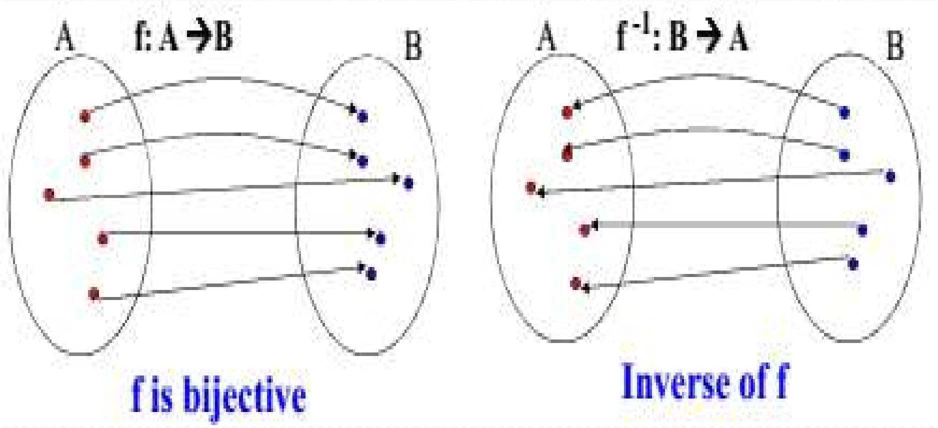
Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

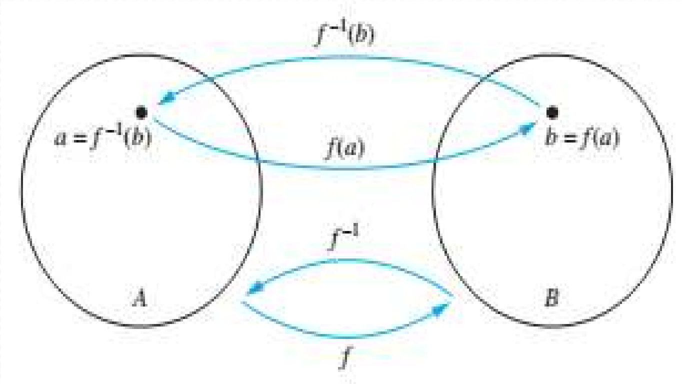
❖ ***To show that f is not surjective***

Find a particular $y \in B$ such that $f(x) = y$ for all $x \in A$.

Type of Function

- **Inverse functions::** Let f be a bijection from set A to set B. The inverse function of f is the function that assigns to an element b from B the unique element a in A such that $f(a) = b$.
- The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$, when $f(a) = b$.
- If the inverse function of f exists, f is called invertible.
- **Note:** If f is not a bijection then it is not possible to define the inverse function of f.





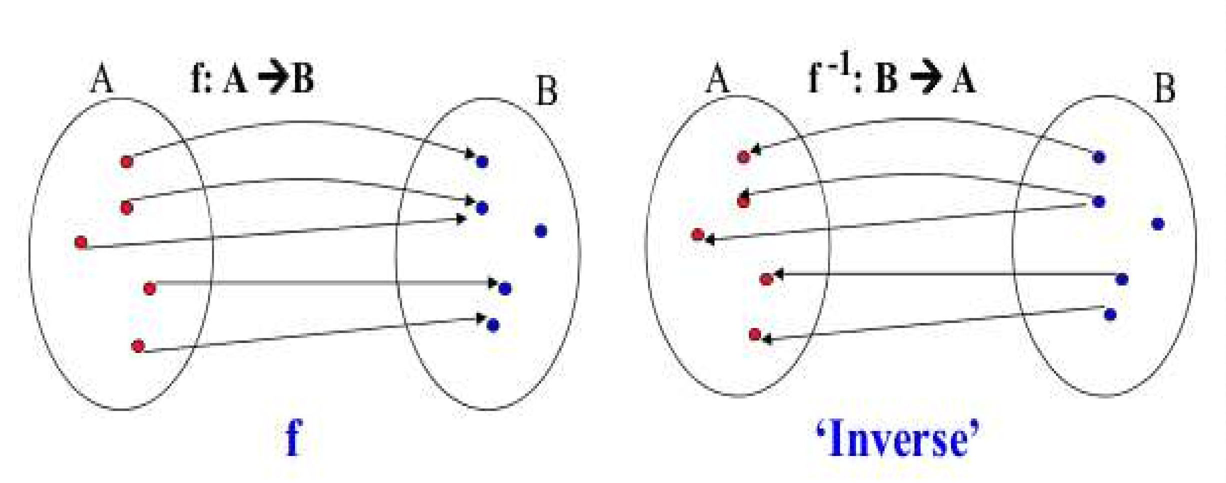
Type of Function

❖ **Inverse functions::**

❖ **Note:** If f is not a bijection then it is not possible to define the inverse function of f.

Solution:

Case 1: Assume f is not one-to-one: Inverse is not a function. One element of B is mapped to two different elements.



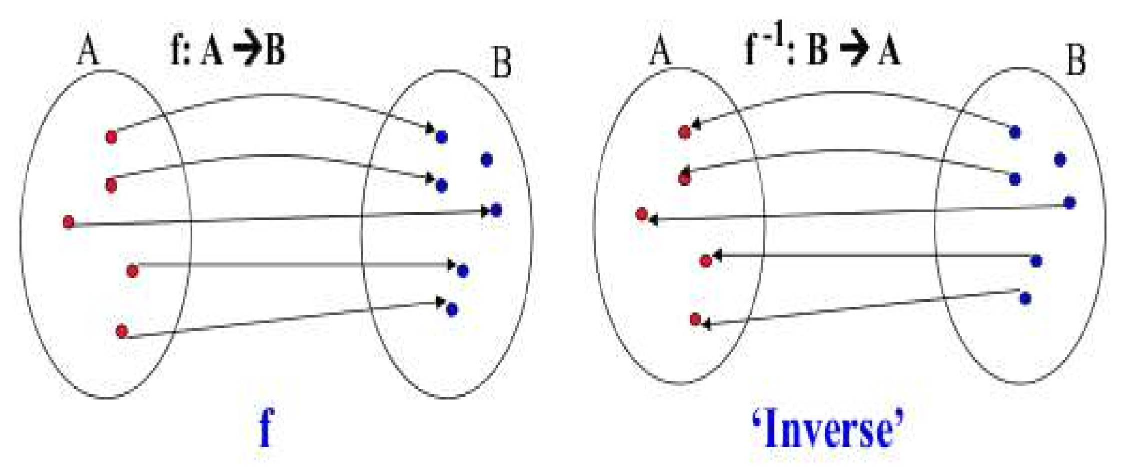
Type of Function

❖ **Inverse functions::**

❖ **Note:** If *f* is not a bijection then it is not possible to define the inverse function of *f*.

Solution:

Case 2: Assume f is not onto: Inverse is not a function. One element of B is not assigned any value in B.



Type of Function

❖ **Example:** Let f be the function from {a, b, c} to {1, 2, 3} such that f(a) = 2, f(b) = 3, and f(c) = 1. Is f invertible, and if it is, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence.

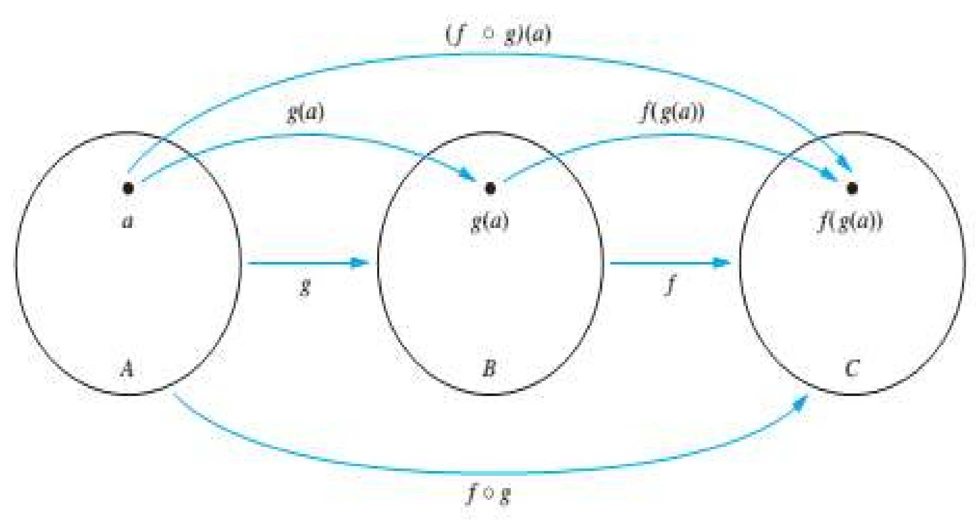
The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Type of Function

❖ **Composition of Functions::**

❖ Let ***g*** be a function from the set A to the set B and let ***f*** be a function from the set B to the set C. The composition of the functions ***f and g***, denoted for all a ∈ A by **f ∘ g**, is defined by

$$(f \circ g)(a) = f(g(a)).$$



Type of Function

- **Example:** Let g be the function from the set {a, b, c} to itself such that g(a) = b, g(b) = c, and g(c) = a.

  Let f be the function from the set {a, b, c} to the set {1, 2, 3} such that f(a) = 3, f(b) = 2, and f(c) = 1.

  What is the composition of $(f \circ g)$, and what is the composition of $(g \circ f)$?

- **Solution:** The composition $f \circ g$ is defined by

$$(f \circ g)(a) = f(g(a)) = f(b) = 2,$$

$$(f \circ g)(b) = f(g(b)) = f(c) = 1, \text{ and}$$

$$(f \circ g)(c) = f(g(c)) = f(a) = 3.$$

  Note that $(g \circ f)$ is not defined, because the range of f is not a subset of the domain of g.

Type of Function

❖ **Example:** Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f ?

Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

and

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$$

❖ Remark: Note that even though $f \circ g$ and $g \circ f$ are defined for the functions f and g in above Example, $f \circ g$ and $g \circ f$ are not equal.

❖ In other words, the commutative law does not hold for the composition of functions.

Type of Function

❖ **Example:** Given $f(x) = x^2 + 6$ and $g(x) = 2x – 1$,

Find a) $(f \circ g)(x)$ and b) $(g \circ f)(x)$

Solution:

a) $(f \circ g)(x)$

$= f(2x – 1)$

$= (2x – 1)^2 + 6$

$= 4x^2 – 4x + 1 + 6$

$= 4x^2 – 4x + 7$

b) $(g \circ f)(x)$

$= g(x^2 + 6)$

$= 2(x^2 + 6) – 1$

$= 2x^2 + 12 – 1$

$= 2x^2 + 11$

Type of Function

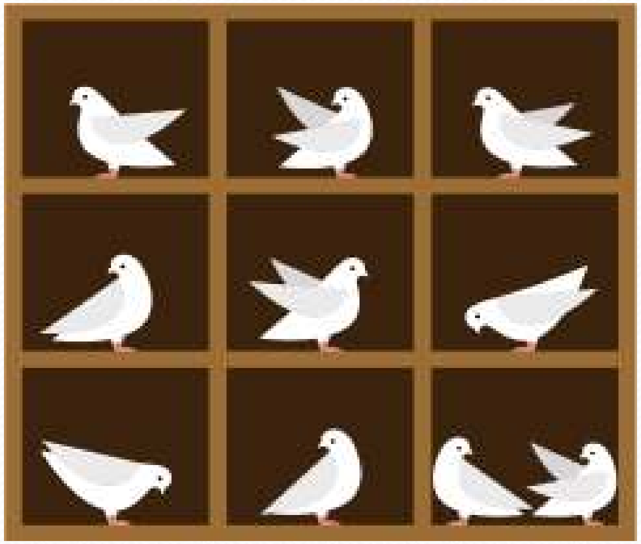
❖ **Example:** Let f(x)= x+2, g(x)= x-2 and h(x)=3x for x ∈R, where R is set of real numbers. Find (gOf), (fOg), (fOf), (gOg), (fOh), (hOg), (hOf),(fOhOg)

Solution:

- ✓ (gOf) = g(f(x)) = g(x+2) = (x+2)-2 = x
- ✓ (fOg) = f(g(x)) = f(x-2) = (x-2)+2 = x
- ✓ (fOf) = f(f(x)) = f(x+2) = (x+2)+2 = x+4
- ✓ (gOg) = g(g(x)) = g(x-2) = (x-2)-2 = x-4
- ✓ (fOh) = f(h(x)) = f(3x) = (3x)+2 = 3x+2
- ✓ (hOg) = h(g(x)) = h(x-2) =3(x-2) = 3x-6
- ✓ (hOf) = h(f(x)) = h(x+2) =3(x+2) = 3x+6
- ✓ (fOhOg) = fOh(g(x)) = fOh(x-2) =f(h(x-2)) = f(3x-6) = (3x-6)+2 = 3x-4

Pigeonhole Principle

- If k is a positive integer and k+1 objects are placed into k boxes, then at least one of the boxes will contain two or more objects. OR
- In mathematics, the pigeonhole principle states that if n items are put into m containers, with n > m, then at least one container must contain more than one item.
  - **Proof:** We prove the pigeonhole principle using a proof by contraposition.
  - Suppose that none of the k boxes contains more than one object.
  - Then the total number of objects would be at most k. This is a contradiction, because there are at least k+1 objects.



Pigeons in holes. Here there are n = 10 pigeons in m = 9 holes. Since 10 is greater than 9, the pigeonhole principle says that at least one hole has more than one pigeon.

Pigeonhole Principle

- The abstract formulation of the principle: Let X and Y be finite sets and let $f: X \to Y$ be a function.
  - If X has more elements than Y, then f is not one-to-one.
  - If X and Y have the same number of elements and f is onto, then f is one-to-one.
  - If X and Y have the same number of elements and f is one-to-one, then f is onto.

- If "A" is the average number of pigeons per hole, where A is not an integer then
  - At least one pigeon hole contains **ceil[A]** (smallest integer greater than or equal to A) pigeons
  - Remaining pigeon holes contains at most **floor[A]** (largest integer less than or equal to A) pigeons.

Pigeonhole Principle

- **Example 01:** In a group of 366 people, there must be two people with the same birthday.
- **Example 02:** In a group of 27 English words, at least two words must start with the same letter.
- **Example 03:** How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

  Solution: There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

Pigeonhole Principle

- **Generalized Pigeon hole Principle:::** If n pigeonholes are occupied by Kn+1 or more pigeons then at least one pigeonhole is occupied by K+1 or more pigeons. OR
- If N objects are placed into k boxes, then there is at least one box containing at least N/k objects.

- **Example 4:** Find the minimum no of students in a class to be ensure that three of them born in the same month.

Solution: n = 12, K+1 =3 i.e. K=2, Kn+1 = 2*12+1 = 25

Pigeonhole Principle

- **Example 5:** What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

  Solution: n = 5, K+1 =6 i.e. K=5, Kn+1 = 5*5+1 = 26

- **Example 6:** Show that 7 colors are used to paint 50 bicycles, and then at least 8 bicycles will be of same color.

  Solution: n = 7, K+1 =8 i.e. K=7, Kn+1 = 7*7+1 = 50

- **Example 7:** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

  Solution: n = 4, K+1 =3 i.e. K=2, Kn+1 = 4*2+1 = 09

Pigeonhole Principle

❖ **Example 8:** If (Kn+1) pigeons are kept in n pigeon holes where K is a positive integer, what is the average no. of pigeons per pigeon hole?

Solution:

✓ Average number of pigeons per hole = (Kn+1)/n = K + 1/n

✓ Therefore at least a pigeonholes contains (K+1) pigeons i.e., ceil[K +1/n] and remaining contain at most K i.e., floor[k+1/n] pigeons.

✓ i.e., the minimum number of pigeons required to ensure that at least one pigeon hole contains (K+1) pigeons is (Kn+1).

Pigeonhole Principle

❖ **Example 9:** A bag contains 10 red marbles, 10 white marbles, and 10 blue marbles. What is the minimum no. of marbles you have to choose randomly from the bag to ensure that we get 4 marbles of same color?

Solution: Apply pigeonhole principle.

- ✓ No. of colors (pigeonholes) n = 3 and No. of marbles (pigeons) K+1 = 4
- ✓ Therefore the minimum no. of marbles required = Kn+1
- ✓ By simplifying we get Kn+1 = 10.
- ✓ Verification: ceil[Average] is [Kn+1/n] = 4
- ✓ [Kn+1/3] = 4
- ✓ Kn+1 = 10
- ✓ i.e., 3 red + 3 white + 3 blue + 1(red or white or blue) = 10

Pigeonhole Principle

- **Example 10:** Show that in any set of six classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day, assuming that no classes are held on weekends.

  Solution: There are six classes: these are the pigeons.

  There are five days on which classes may meet (Monday through Friday): these are the pigeonholes.

  Each class must meet on a day (each pigeon must occupy a pigeonhole).

  By the pigeonhole principle at least one day must contain at least two classes.

Pigeonhole Principle

❖ **Example 11:** What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

Solution: The pigeons are the students (no slur intended), and the pigeonholes are the states, 50 in number.

By the generalized pigeonhole principle:

$n = 50, K+1 = 100$ i.e. $K = 99$, $Kn+1 = 99*50+1 = 4951$