SE (Comp.Engg.)

Unit II

Discrete Structures

<u>Relations and Functions</u>

Cartesian products

The Cartesian product of set A and set B is denoted by $A \times B$ and equals $\{(a, b) \mid a \in A \text{ and } b \in B\}$.

The elements of $A \times B$ are ordered pairs.

The elements of $A_1 \times A_2 \times \ldots \times A_n$ are ordered n-tuples.

$|A \times B| = |A| \times |B|$

- Ex . A={2, 3, 4}, B={4, 5} , C={x,y}
- A × B ={<2,4>,<2,5>,<3,4>,<3,5>,<4,4>,<4,5>}

Relations

- Any subsets of A×B is called
  a binary relation from A to B.
  Any subset of A×A is called a binary relation on A.
  For finite sets A and B with |A|=m and |B|=n, there are 2^{mn} relations from A to B.

- **Example:** Let A = {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a < b} ?
- **Solution:** R = {(1,2),(1,3),(1,4),(2,3)(2,4)(3,4)}

- **Domain= set of first elements in the cartesian product .**
- **Range= set of second elements in the cartesian product .**

Domain={1,2,3}
Range={2,3,4}

- **Converse** of a Relation A is given by the relation Ã such that the elements in the ordered pairs in A are interchanged.
- i.e if xAy then y Ã x.

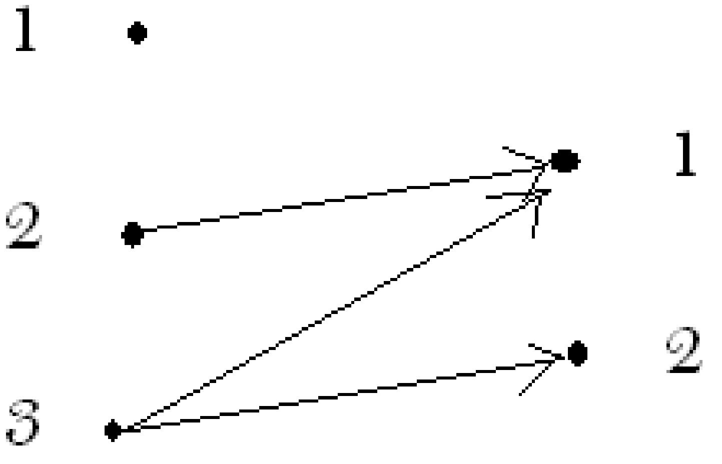
Matrix Representation of a Relation

- $M_R = [m_{ij}]$ (where i=row, j=col)
  - m_{ij}={1 iff $(i,j) \in R$ and 0 iff $(i,j) \notin R$}

- Ex: R : {1,2,3}→{1,2} where x > y
  - R = {(2,1),(3,1),(3,2)}

	1	2
1	0	0
2	1	0
3	1	1

Graph Representation of a Relation



Properties of Relations

- A relation R on a set A is called **reflexive** if
  $(a, a) \in R$ for every element $a \in A$.
- A relation on a set A is called **irreflexive** if
  $(a, a) \notin R$ for every element $a \in A$.

- A relation R on a set A is called **symmetric** if
  (b, a)∈R whenever (a, b)∈R for all a, b∈A.

A relation R on a set A is called **asymmetric** if

- (a, b)∈R implies that (b, a)∉R for all a,b∈A.
- A relation R on a set A is called **antisymmetric** if
  whenever (a, b)∈R and (b, a)∈R, a = b

- A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

Equivalence Relations

- Any binary relation that is:

  Reflexive
  Symmetric
  Transitive

Is Equivalence Relations

Equivalence Classes

Let R be an equivalence relation on a set A. The set of all elements that are related to an element **a** of A is called the **equivalence class** of **a**.

- The equivalence class of a with respect to R is denoted by $\mathbf{[a]_R}$.
- If $\mathbf{b \in [a]_R}$, b is called a **representative** of this equivalence class.

Ex. A={1,2,3}
R={(1,2)(2,3)}
[a]={x is element of A such that (x,a) is element in R}

$[1]_R$ ={} [2]={1} [3]={2}

Partition

A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i,
$i \in I$, forms a partition of S if and only if

- (i) $A_i \neq \emptyset$ for $i \in I$
- $A_i \cap A_j = \emptyset$, if $i \neq j$
- $\cup_{i \in I} A_i = S$

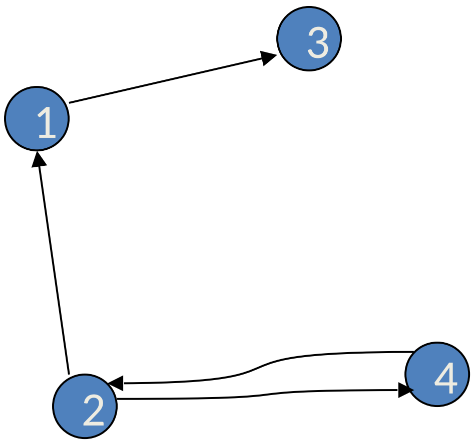
S={1,2,3...........8,9} check for each of following partition or not....

{{1,3,5}{2,6}{4,8,9}} not partitions as 7 is not in any of the subset

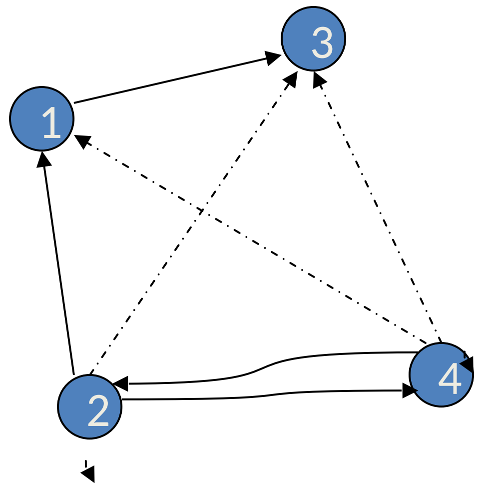
{{1,3,5}{2,4,6,8}{7,9}} valid partitions

{{1,3,5}{2,4,6,8}{5,7,9}} not partitions as {1,3,5} {5,7,9} are not disjoint

Warshall's algorithm to find Transitive Closure for given Graph



Graph 1 given



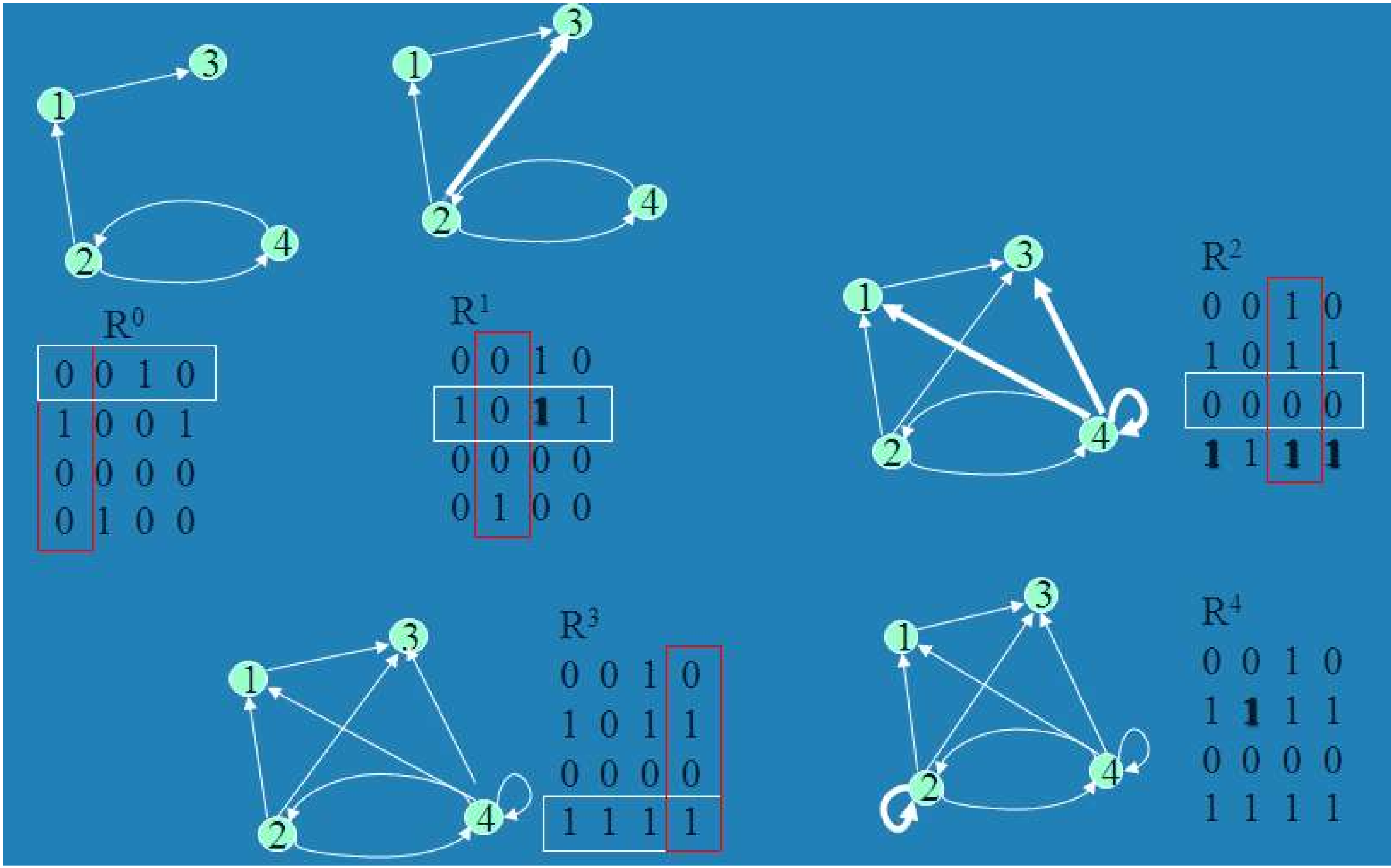
Transitive Closure of Graph 1

C1 R2=R2 OR R1,

C2 R4=R4 OR R2,

C3

C4 R2=R2 OR R4, R4=R4 OR R4



Partial order Relation

- A relation *R* on a set *S* is called a partial ordering or partial order if it is **reflexive, antisymmetric, and transitive.**
- A set *S* together with a partial ordering *R* is called a partially ordered set, or **POSET** and denoted by (*S*,*R*). A partial order R is also denoted as . $(R, \leq$

- The elements a and b of a poset $(S, \preccurlyeq)$ are called **comparable** if either $a \preccurlyeq b$ or $b \preccurlyeq a$. Otherwise a and b are called **incomparable**.

- If $(S, \preccurlyeq)$ is a partial ordering set and every two elements of *S* are comparable, *S* is called a **totally ordered** or **linearly ordered** set.
- A totally ordered set is called a **Chain.**

Hasse Diagrams

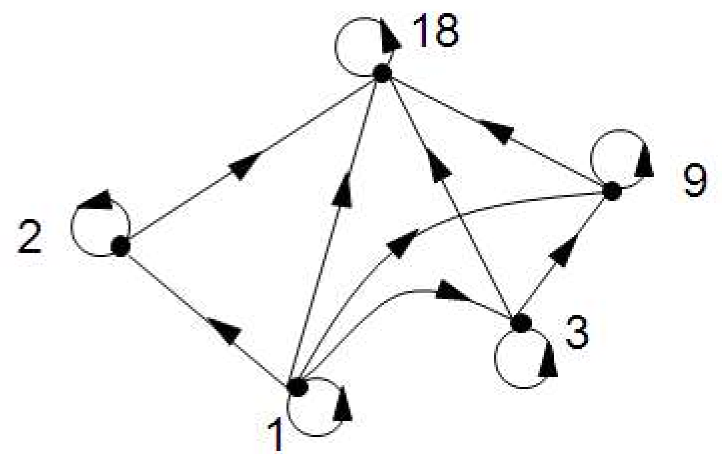
- Given any partial order relation defined on a finite set, it is possible to draw the directed graph so that all of these properties are satisfied.
- This makes it possible to associate a somewhat simpler graph, called a ***Hasse diagram***, with a partial order relation defined on a finite set.

- Start with a directed graph of the relation in which all arrows point upward. Then eliminate:

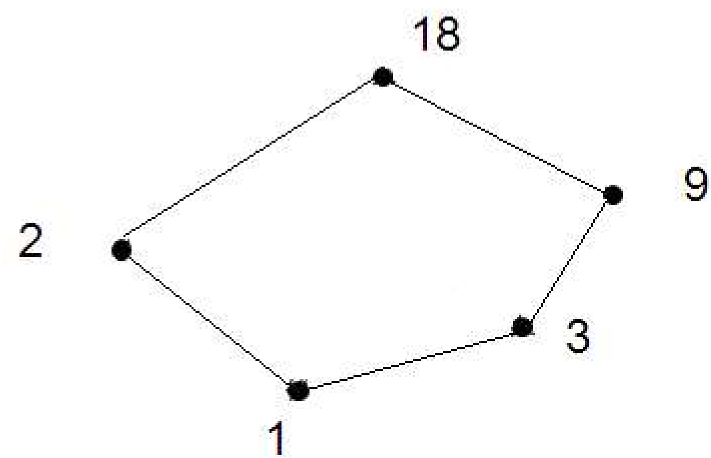
1. the loops at all the vertices,
2. all arrows whose existence is implied by the transitive property,
3. the direction indicators on the arrows.

- Let A = {1, 2, 3, 9, 19} and consider the “divides” relation on A:
- For all

a|b or b=Ka for some integer K

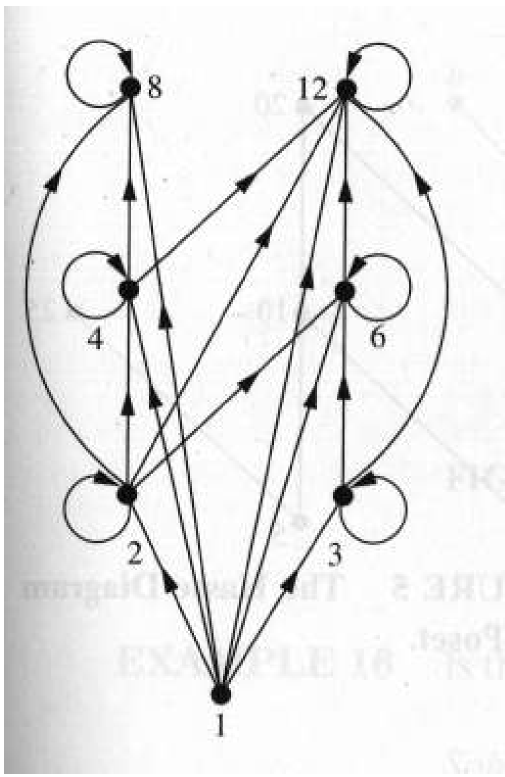


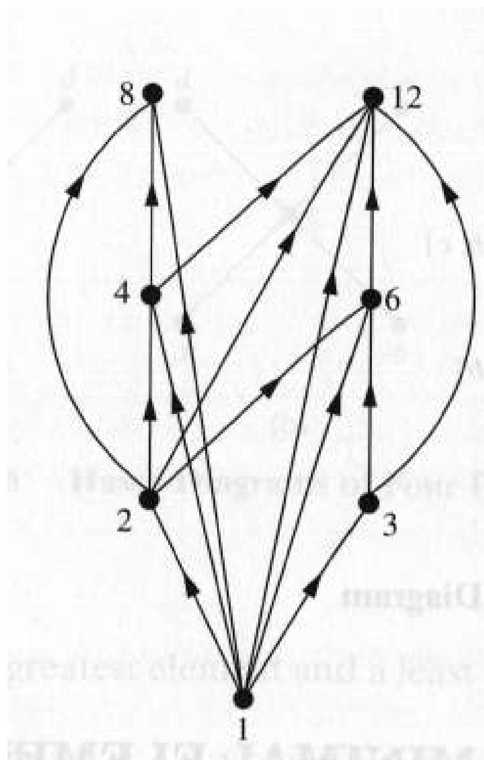
Dighraph

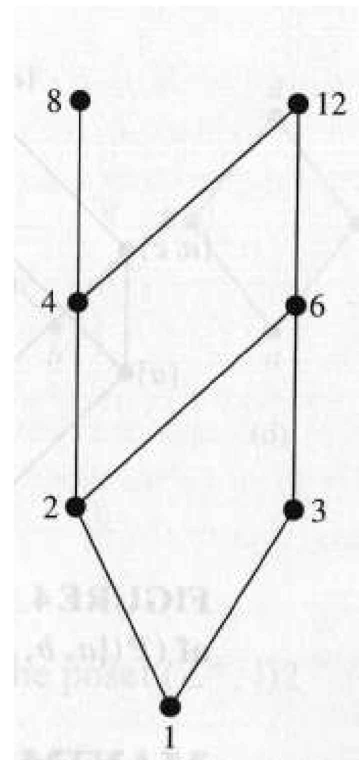


Hasse Diagram

- For the poset ({1,2,3,4,6,8,12}, |)







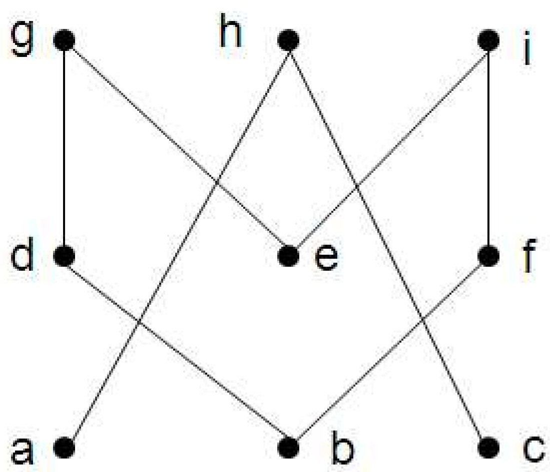
Extremal Elements: Maximal

- An element a in **a** poset (S, ≤) is called **maximal** if no element **b** in S exists such that,
  $a \leq b$
- If there is one unique maximal element **a**, it is called the maximum element (or the greatest element)

Extremal Elements: Minimal

- An element a in **a** poset (S, ≤) is called **minimal** if no element **b** in S exists such that,

  $b \leq a$
- If there is one unique minimal element **a**, it is called the minimum element (or the least element)



g
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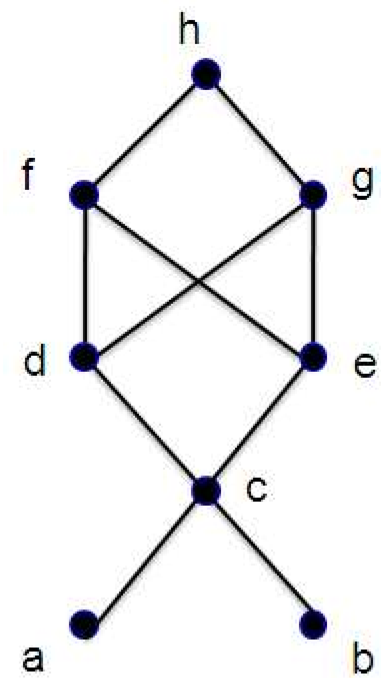

h
f
g
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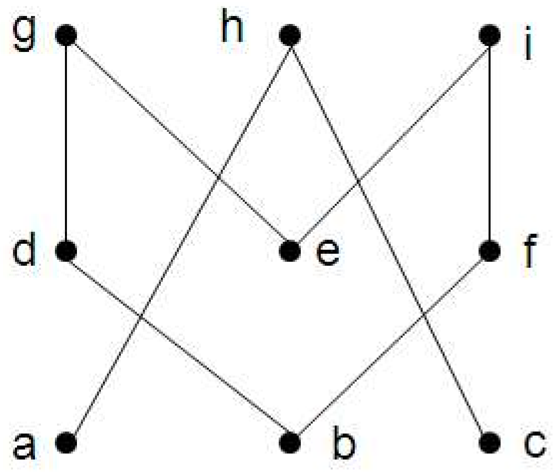
c
a
b

- Let (S, ≤) be a poset and let A⊆S. If u is an element of S such that a ≤ u for all a∈A then u is an upper bound of A
- An element x that is an upper bound on a subset A and is less than all other upper bounds on A is called the least upper bound on A. We abbreviate it as lub.

- **Definition**: Let (S, ≤) be a poset and let A⊆S. If l is an element of S such that l ≤ a for all a∈A then l is an lower bound of A
- An element x that is a lower bound on a subset A and is greater than all other lower bounds on A is called the greatest lower bound on A. We abbreviate it glb.



Give lower/upper bounds & glb/lub of the sets:

{d,e,f}, {a,c} and {b,d}

{d,e,f}

- Lower bounds: ∅, thus no glb
- Upper bounds: ∅, thus no lub

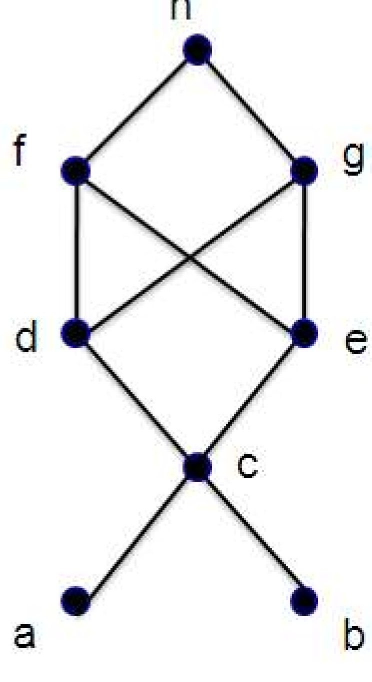
{a,c}

- Lower bounds: ∅, thus no glb
- Upper bounds: {h}, lub: h

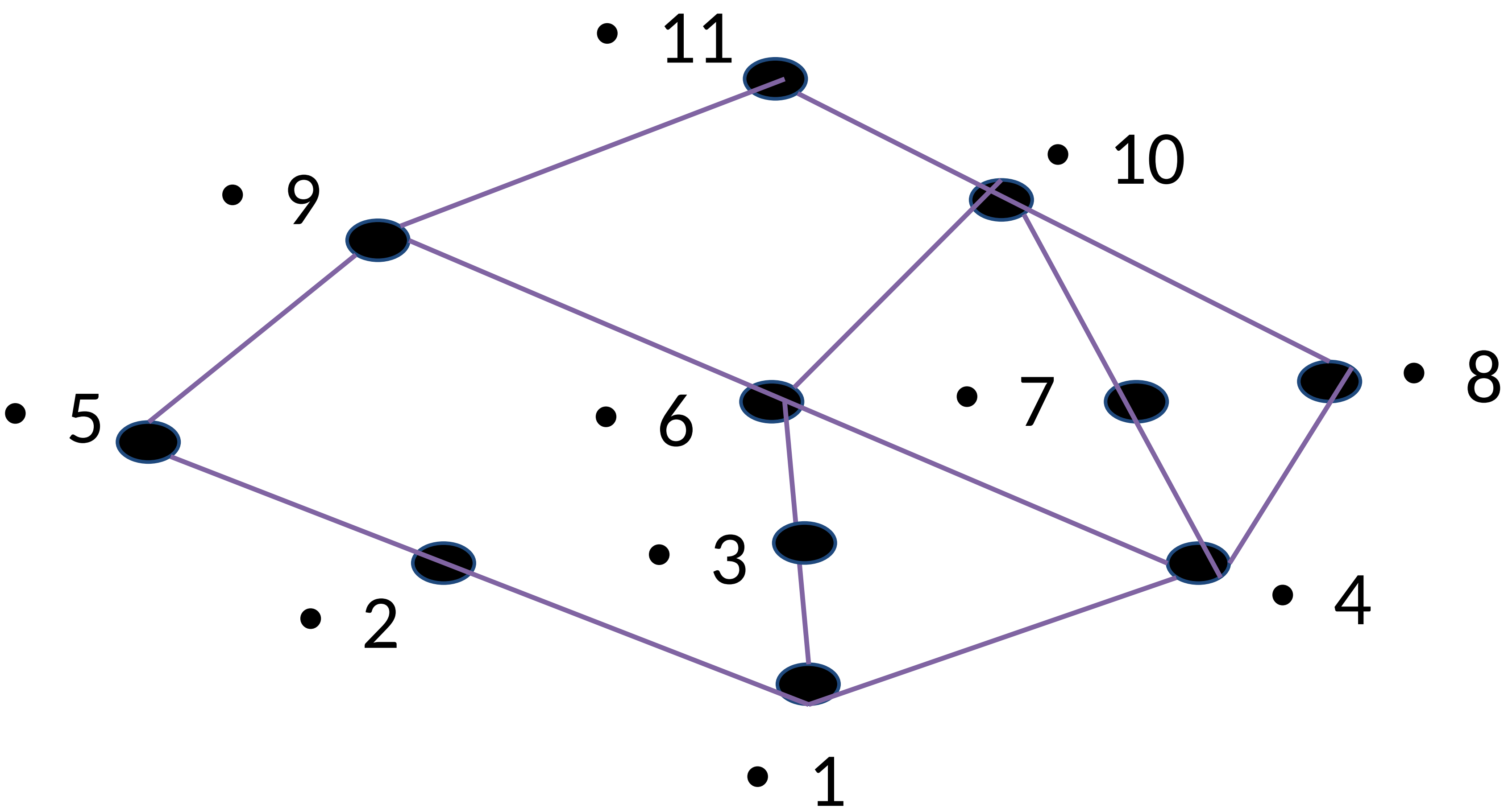
{b,d}

- Lower bounds: {b}, glb: b
- Upper bounds: {d,g}, lub: d because $d \leq g$

- Find all upper and lower bounds of the following subset of A: $B_1=\{a, b\}$; $B_2=\{c, d, e\}$;



Find the LUB and GLB of B={6,7,10} for the following Hasse diagram.



Lattices

- A **lattice** is a partially ordered set in which every pair of elements has both
  - a least upper bound and
  - a greatest lower bound