Unit I: Set Theory and Logic (07 Hours)

- **Introduction** and Significance of Discrete Mathematics
- **Sets**–Naïve Set Theory (Cantorian Set Theory), Axiomatic Set Theory, Set Operations, Cardinality of set, Principle of inclusion and exclusion.
- **Types of Sets**–Bounded and Unbounded Sets, Diagonalization argument, Countable and Uncountable Sets, Finite and Infinite Sets, Countably Infinite and Uncountably Infinite Sets, Power set,
- **Propositional Logic-** Logic, Propositional Equivalences, Application of Propositional Logic-Translating English Sentences,
- Proof by Mathematical Induction and Strong Mathematical Induction
- **Exemplar/ Case Studies:** Know about the great philosophers- Georg Cantor, Richard Dedekind and Aristotle.

❖ INTRODUCTION:

- Mathematics can be broadly classified into two categories −
- **Continuous Mathematics** − It is based upon continuous number line or the real numbers. It is characterized by the fact that between any two numbers, there are almost always an infinite set of numbers. For example, a function in continuous mathematics can be plotted in a smooth curve without breaks.
- **Discrete Mathematics** − It involves distinct values; i.e. between any two points, there are a countable number of points. For example, if we have a finite set of objects, the function can be defined as a list of ordered pairs having these objects, and can be presented as a complete list of those pairs.
- **Discrete Mathematics** is a branch of mathematics involving discrete elements that uses algebra and arithmetic. OR
- **Discrete math =** study of the discrete structures used to represent discrete objects.
- **Discrete objects** are those which are separated from (distinct from) each other.
- **Example:** Integers (aka whole numbers), rational numbers (ones that can be expressed as the quotient of two integers), automobiles, houses, people etc. are all discrete objects.
- On the other hand real numbers which include irrational as well as rational numbers are not discrete.
- As you know between any two different real numbers there is another real number different from either of them. So they are packed without any gaps and cannot be separated from their immediate neighbors.

❖ SET:

❖ Definition

- A set is an unordered collection of different elements. A set can be written explicitly by listing its elements using set bracket. If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.
- Sets are used to group objects together. Often, but not always, the objects in a set have similar properties.
- We write **$a \in A$** to denote that a is an element of the set A. The notation **$a \notin A$** denotes that a is not an element of the set A.
- It is common for sets to be denoted using uppercase letters. Lowercase letters are usually used to denote elements of sets.
- Example of Sets:
  1. A set of all positive integers
  2. A set of all the planets in the solar system
  3. A set of all the states in India
  4. A set of all the lowercase letters of the alphabet

❖ Basic Properties of Sets

- The change in order of writing the elements does not make any changes in the set.
- If one or many elements of a set are repeated, the set remains the same.

❖ Representation of a Set

- Sets can be represented in two different ways −
  1. Roster or Tabular Form
  2. Set Builder Notation

1. Roster or Tabular Form

- The set is represented by listing all the elements comprising it. The elements are enclosed within braces and separated by commas.
- Example 1 − Set of vowels in English alphabet, $A = \{a,e,i,o,u\}$
- Example 2 − Set of odd numbers less than 10, $B = \{1,3,5,7,9\}$

2. Set Builder Notation

- The set is defined by specifying a property that elements of the set have in common.
- The set is described as $A=\{x:p(x)\}$

- Example 1 − The set {a,e,i,o,u} is written as −

  A={x:x is a vowel in English alphabet}
- Example 2 − The set {1,3,5,7,9} is written as −

  B={x:1≤x<10 and (x%2)≠0}
- Example 3 − O = {x | x is an odd positive integer less than 10}, or, specifying the universe as the set of positive integers, as

  O = {x ∈ Z + | x is odd and x < 10}.

❖ Some Important Sets:

- **N** − the set of all natural numbers = {1, 2, 3, 4,.....}
- **Z** − the set of all integers = {.....,−3,−2,−1,0,1,2,3,.....}
- **Z**$^+$ − the set of all positive integers.
- **Q** − the set of all rational numbers.
- **R** − the set of all real numbers.
- **R**$^+$ − the set of positive real numbers.
- **W** − the set of all whole numbers.
- **C** − the set of complex numbers.

❖ Cardinality of a Set

- Cardinality of a set S, denoted by |S|, is the number of elements of the set. The number is also referred as the cardinal number. If a set has an infinite number of elements, its cardinality is ∞.
- Example − |{1,4,3,5}| = 4, |{1,2,3,4,5,…}| = ∞

❖ Cardinal Properties of Sets

- If A and B are finite sets, then n(A ∪ B) = n(A) + n(B) - n(A ∩ B)
- If A ∩ B = ф , then n(A ∪ B) = n(A) + n(B)
- n(A ∩ B) = n(A) + n(B) - n(A ∪ B)
- n(A - B) = n(A) - n(A ∩ B)
- n(B - A) = n(B) - n(A ∩ B)

❖ Type of Set

1. Finite Set

- A set which contains a definite number of elements is called a finite set. Empty set is also called a finite set.

- For Example:
  - ✓ S={x | x ∈ N and 70 > x > 50}
  - ✓ The set of all colors in the rainbow.
  - ✓ N = {x : x ∈ N, x < 7}
  - ✓ P = {2, 3, 5, 7, 11, 13, 17, ... 97}

2. Infinite Set

- A set which contains infinite number of elements is called an infinite set.
- i.e set containing never-ending elements is called an infinite set.
- For example:
  - ✓ A = {x : x ∈ N, x > 1}
  - ✓ B = {x : x ∈ W, x = 2n}
  - ✓ S = {x | x ∈ N and x >10}
  - ✓ Set of all points in a plane
  - ✓ Set of all prime numbers

3. Subset

- A set X is a subset of set Y (Written as X⊆Y) if every element of X is an element of set Y.
- Every set is a subset of itself, i.e., A ⊂ A, B ⊂ B.
- Empty set is a subset of every set.
- Example 1 − Let, Y = {1, 2, 3, 4, 5, 6} and X = {1, 2}. Here set X is a subset of set Y as all the elements of set X is in set Y. Hence, we can write X⊆Y.
- Example 2 − Let, X = {1, 2, 3} and Y = {1, 2, 3}. Here set Y is a subset (Not a proper subset) of set X as all the elements of set Y is in set X. Hence, we can write Y⊆X.

4. Proper Subset

- The term "proper subset" can be defined as **"subset of but not equal to".** A Set X is a proper subset of set Y (Written as X⊂Y) if every element of X is an element of set Y and |X|<|Y|.
- No set is a proper subset of itself.
- Null set or ∅ is a proper subset of every set.
- Example − Let, X={1,2,3,4,5,6} and Y={1,2}. Here set Y⊂X since all elements in Y are contained in X too and X has at least one element is more than set Y.

5. Super Set

- Whenever a set X is a subset of set Y, we say the Y is a superset of X and written as Y ⊇ X.

- For Example X = {a, e, i, o, u} and Y = {a, b, c, ..., z}
- Here X ⊆ Y i.e., X is a subset of Y but Y ⊇ X i.e., Y is a super set of X.

6. Universal Set

- It is a collection of all elements in a particular context or application. All the sets in that context or application are essentially subsets of this universal set. Universal sets are represented as U.
- Example-
  - ✓ We may define U as the set of all animals on earth. In this case, set of all mammals is a subset of U, set of all fishes is a subset of U, set of all insects is a subset of U, and so on.
  - ✓ If A = {1, 2, 3} B = {2, 3, 4} C = {3, 5, 7} then U = {1, 2, 3, 4, 5, 7} [Here A ⊆ U, B ⊆ U, C ⊆ U and U ⊇ A, U ⊇ B, U ⊇ C]
  - ✓ If P is a set of all whole numbers and Q is a set of all negative numbers then the universal set is a set of all integers.
  - ✓ If A = {a, b, c}, B = {d, e} and C = {f, g, h, i} then U = {a, b, c, d, e, f, g, h, i} can be taken as universal set.

7. Empty Set or Null Set

- A set which does not contain any element is called an empty set, or the null set or the void set and it is denoted by ∅ and is read as phi.
- In roster form, ∅ is denoted by {}.
- An empty set is a finite set, since the number of elements in an empty set is finite, i.e., 0.
- Example - S = {x | x ∈ N and 7 < x < 8} = ∅

8. Singleton Set or Unit Set

- Singleton set or unit set contains only one element. A singleton set is denoted by {S}.
- Example-
  - ✓ S = { x | x ∈ N, 7 < x < 9 } = {8}
  - ✓ Let A = {x : x ∈ N and x^2 = 4}

    Here A is a singleton set because there is only one element 2 whose square is 4.
  - ✓ Let B = {x : x is a even prime number}

    Here B is a singleton set because there is only one prime number which is even, i.e., 2.

9. Equal Set

- If two sets contain the same elements they are said to be equal.
- Example -
  - ✓ If A={1,2,6}and B={6,1,2}, they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

10. Equivalent Set

- If the cardinalities of two sets are same, they are called equivalent sets.
- The symbol for denoting an equivalent set is '↔'.
- Example − If A={1,2,6} and B={16,17,22} they are equivalent as cardinality of A is equal to the cardinality of B. i.e. $|A| = |B| = 3$. Therefore $A \leftrightarrow B$.

11. Disjoint Set

- Two sets A and B are said to be disjoint, if they do not have any element in common. OR
- Two sets are called disjoint if their intersection is the empty set.
- Disjoint sets have the following properties:
  - ✓ $n(A\cap B) = \emptyset$
  - ✓ $n(A\cup B) = n(A) + n(B)$
- Example -
  - ✓ A = {1,2,6} and B={7,9,14}, A = {x : x is a prime number} and B = {x : x is a composite number}. Here A and B do not have any element in common and are disjoint sets.
  - ✓ Let A = {1, 3, 5, 7, 9} and B = {2, 4, 6, 8, 10}. Because $A \cap B = \emptyset$, A and B are disjoint.

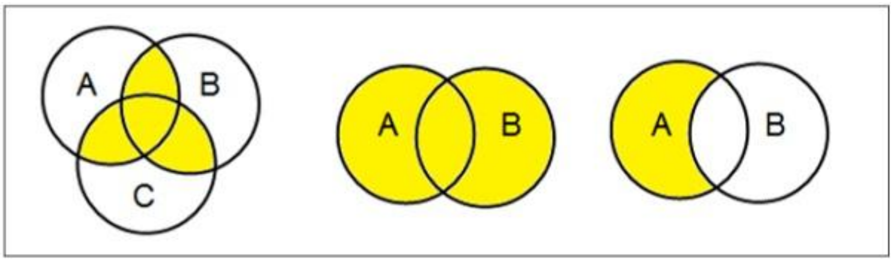
12. Overlapping sets

- Two sets A and B are said to be overlapping if they contain at least one element in common.
- Example -
  - ✓ A = {a, b, c, d} and B = {a, e, i, o, u}, Here common element 'a'.
  - ✓ Let, A = {1, 2, 6} and B = {6, 12, 42}. There is a common element '6'; hence these sets are overlapping sets.
  - ✓ $X = \{x : x \in N, x < 4\}$ and $Y = \{x : x \in I, -1 < x < 4\}$. Here, the two sets contain three elements in common, i.e., (1, 2, 3).

❖ **Venn Diagrams**

- Venn diagram, invented in 1880 by John Venn, is a schematic diagram that shows all possible logical relations between different mathematical sets.

- A Venn diagram (also called **primary diagram, set diagram or logic diagram**) is a diagram that shows all possible logical relations between a finite collections of different sets.
- These diagrams depict elements as points in the plane, and sets as regions inside closed curves.
- A Venn diagram consists of multiple overlapping closed curves, usually circles, each representing a set.
- The points inside a curve labelled S represent elements of the set S, while points outside the boundary represent elements not in the set S.
  - Venn diagrams are used to illustrate various operations like union, intersection and difference.



❖ Set Operation

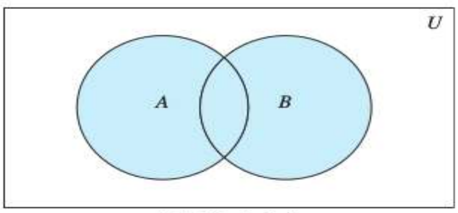
1. Union

- Let A and B be sets. The union of the sets A and B, denoted by A ∪ B, is the set that contains those elements that are either in A or in B, or in both.
- An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B.

$$\mathbf{A \cup B = \{x \mid x \in A \vee x \in B\}}$$

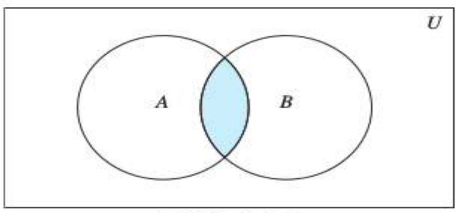
- Example − If A={10,11,12,13} and B = {13,14,15}, then

  A∪B={10,11,12,13,14,15}.
- Example −The union of the sets {1, 3, 5} and {1, 2, 3} is the set {1, 2, 3, 5}; that is, {1, 3, 5} ∪ {1, 2, 3} = {1, 2, 3, 5}.



$A \cup B$ is shaded.

FIGURE 1 Venn Diagram of the Union of *A* and *B*.



$A \cap B$ is shaded.

FIGURE 2 Venn Diagram of the Intersection of *A* and *B*.

2. Interaction

- Let A and B be sets. The intersection of the sets A and B, denoted by A ∩ B, is the set containing those elements in both A and B.
- An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B.

$$\mathbf{A \cap B = \{x \mid x \in A \land x \in B\}.}$$

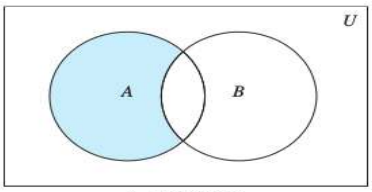
- If A={11,12,13} and B={13,14,15}, then A∩B={13}.
- The intersection of the sets {1, 3, 5} and {1, 2, 3} is the set {1, 3}; that is,{1, 3, 5} ∩ {1, 2, 3} = {1, 3}.

3. Difference of Sets (Relative Complement)

- Let A and B be sets. The difference of A and B, denoted by A − B, is the set containing those elements that are in A but not in B.
- The difference of A and B is also called the complement of B with respect to A.
- The difference of sets A and B is sometimes denoted by A\B.
- An element x belongs to the difference of A and B if and only if x ∈ A and x ∉ B.

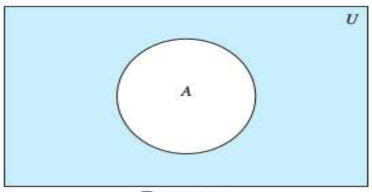
$$\mathbf{A - B = \{x \mid x \in A \land x \notin B\}.}$$

- Example - The difference of {1, 3, 5} and {1, 2, 3} is the set {5}; that is, {1, 3, 5} − {1, 2, 3} = {5}. This is different from the difference of {1, 2, 3} and {1, 3, 5}, which is the set {2}.



A − B is shaded.

FIGURE 3 **Venn Diagram for the Difference of *A* and *B*.**



$\bar{A}$ is shaded.

FIGURE 4 **Venn Diagram for the Complement of the Set *A*.**

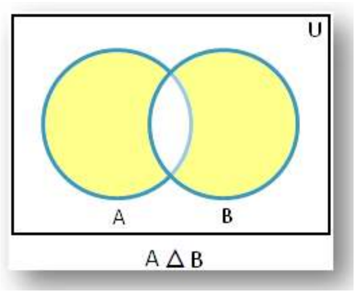
4. Complement of a Set

- Let U be the universal set. The complement of the set A, denoted by **Ā**, is the complement of A with respect to U. Therefore, the complement of the set A is U − A.
- An element belongs to **Ā** if and only if x ∉ A. This tells us that
  - $\mathbf{\bar{A} = \{x \in U \mid x \notin A\}}$
- Let A = {a, e, i, o, u} (where the universal set is the set of letters of the English alphabet). Then A = {b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z}.

- Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

5. Symmetric Difference

- Let A and B are two sets. The symmetric difference of two sets A and B is the set (A – B) ∪ (B – A) and is denoted by A ∆ B.
- Thus, A ∆ B = (A – B) ∪ (B – A) = {x : x ∉ A ∩ B}
- or, A ∆ B = {x : [x ∈ A and x ∉ B] or [x ∈ B and x ∉ A]}.
- A ∆ B = {x | x ∈ A –B ∨ B-A}



- Example - If A = {1, 2, 3, 4, 5, 6, 7, 8} and B = {1, 3, 5, 6, 7, 8, 9}, then A – B = {2, 4}, B – A = {9} and A ∆ B = {2, 4, 9}.
- Example - If P = {a, c, f, m, n} and Q = {b, c, m, n, j, k} then P ∆ Q = {a, b, f, j, k}

❖ Cartesian Products

- The order of elements in a collection is often important. Because sets are unordered, a different structure is needed to represent ordered collections. This is provided by ordered n-tuples.
- The **ordered n-tuple** $(a_1, a_2. . . a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, . . . , and an as its n^{th} element.
- Let A and B be sets. The Cartesian product of A and B, denoted by A × B, is the set of all ordered pairs (a, b), where a ∈ A and b ∈ B. Hence,

  A × B = {(a, b) | a ∈ A ∧ b ∈ B}.
- Example - What is the Cartesian product of A = {1, 2} and B = {a, b, c}?

  A × B = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)}.

  B × A = {(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)}
- Note that the Cartesian products A×B and B×A are not equal, unless A = ∅ or B = ∅.
- **Cardinality of the Cartesian product:**

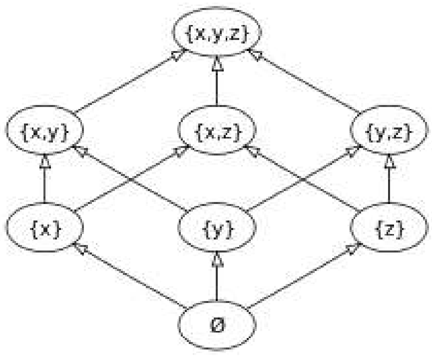
  Let A and B be sets. The **Cardinality** of the Cartesian product of A and B, denoted by |A x B| = |A| * |B|.

❖ Set Identities

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

❖ Power Sets

- Given a set S, the power set of S is the set of all subsets of the set S, including the empty set and S itself. The power set of S is denoted by P (S).
- If a set has n elements, then its power set has 2^n elements.



- If S is the set {x, y, z}, then the subsets of S are
  {}, {x}, {y}, {z}, {x, y}, {x, z}, {y, z}, {x, y, z}
  and hence the power set of S is {{}, {x}, {y}, {z}, {x, y}, {x, z}, {y, z}, {x, y, z}}.

- Example - What is the power set of the set {0, 1, 2}?

  Solution: The power set P ({0, 1, 2}) is the set of all subsets of {0, 1, 2}. Hence,

  $$P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

- Example - What is the power set of the empty set? And

  What is the power set of the set {∅}?

  Solution: The empty set has exactly one subset, namely, itself. Consequently,

  $$P(\emptyset) = \{\emptyset\}.$$

  The set {∅} has exactly two subsets, namely, ∅ and the set {∅} itself. Therefore,

  $$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$

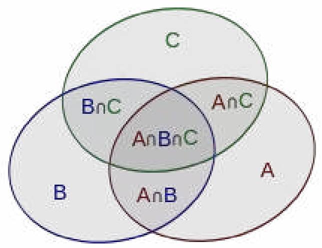
❖ Principle of Inclusion and Exclusion

- The inclusion–exclusion principle is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets; symbolically expressed as

  $$|A \cup B| = |A| + |B| - |A \cap B|$$

- The principle is more clearly seen in the case of three sets, which for the sets A, B and C is given by

  $$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



- This formula can be verified by counting how many times each region in the Venn diagram figure is included.
- In this case, when removing the contributions of over-counted elements, the number of elements in the mutual intersection of the three sets has been subtracted too often, so must be added back in to get the correct total.
- Generalizing the results of these examples gives the principle of inclusion–exclusion.
- To find the cardinality of the union of n sets:
  - ✓ Include the cardinalities of the sets.
  - ✓ Exclude the cardinalities of the pairwise intersections.
  - ✓ Include the cardinalities of the triple-wise intersections.
  - ✓ Exclude the cardinalities of the quadruple-wise intersections.
  - ✓ Include the cardinalities of the quintuple-wise intersections.

- ✓ Continue, until the cardinality of the n-tuple-wise intersection is included (if n is odd) or excluded (n even).

- In its general form, the principle of inclusion–exclusion states that for finite sets A1, ..., An, one has the identity:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \cdots$$
$$\cdots + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|.$$

- **Example:**

1. Let A and B be two finite sets such that n(A) = 20, n(B) = 28 and n(A ∪ B) = 36, find n(A ∩ B).

Solution: Using the formula n(A ∪ B) = n(A) + n(B) - n(A ∩ B).

then n(A ∩ B) = n(A) + n(B) - n(A ∪ B)

= 20 + 28 - 36

= 48 - 36

= 12

2. If n(A - B) = 18, n(A ∪ B) = 70 and n(A ∩ B) = 25, then find n(B).

Solution: Using the formula n(A∪B) = n(A - B) + n(A ∩ B) + n(B - A)

70 = 18 + 25 + n(B - A)

70 = 43 + n(B - A)

n(B - A) = 70 - 43

n(B - A) = 27

Now n(B) = n(A ∩ B) + n(B - A) = 25 + 27 = 52

3. In a group of 60 people, 27 like cold drinks and 42 like hot drinks and each person likes at least one of the two drinks. How many like both coffee and tea?

Solution: Let A = Set of people who like cold drinks.

B = Set of people who like hot drinks.

Given: (A ∪ B) = 60 n(A) = 27 n(B) = 42 then;

n(A ∩ B) = n(A) + n(B) - n(A ∪ B)

= 27 + 42 - 60

= 69 - 60 = 9

Therefore, 9 people like both tea and coffee.

4. In a class of 40 students, 15 like to play cricket and football and 20 like to play cricket. How many like to play football only but not cricket?

Solution: Let C = Students who like cricket And F = Students who like football

$C \cap F$ = Students who like cricket and football both

C - F = Students who like cricket only

F - C = Students who like football only.

$n(C) = 20$ $n(C \cap F) = 15$ $n(C \cup F) = 40$ $n(F) = ?$

$n(C \cup F) = n(C) + n(F) - n(C \cap F)$

$40 = 20 + n(F) - 15$

$40 = 5 + n(F)$

$40 - 5 = n(F)$ Therefore, $n(F) = 35$

Therefore, $n(F - C) = n(F) - n(C \cap F) = 35 - 15 = 20$

Therefore, Number of students who like football only but not cricket = 20

5. In a survey of university students, 64 had taken mathematics course, 94 had taken chemistry course, 58 had taken physics course, 28 had taken mathematics and physics, 26 had taken mathematics and chemistry, 22 had taken chemistry and physics course, and 14 had taken all the three courses. Find how many had taken one course only.

Solution:

Step 1 :Let M, C, P represent sets of students who had taken mathematics, chemistry and physics respectively

Step 2 : From the given information, we have

$n(M) = 64$, $n(C) = 94$, $n(P) = 58$,

$n(M \cap P) = 28$, $n(M \cap C) = 26$,

$n(C \cap P) = 22$, $n(M \cap C \cap P) = 14$

Step 3 :No. of students who had taken only Math

$= n(M) - [n(M \cap P) + n(M \cap C) - n(M \cap C \cap P)]$

$= 64 - [28+26-14]$

$= 64 - 40$

$= 24$

Step 4 :No. of students who had taken only Chemistry

$= n(C) - [n(M \cap C) + n(C \cap P) - n(M \cap C \cap P)]$

$= 94 - [26+22-14]$

$= 94 - 34$

= 60

Step 5 : No. of students who had taken only Physics

= n(P) - [n(M∩P) + n(C∩P) - n(M∩C∩P)]

= 58 - [28+22-14]

= 58 - 36

= 22

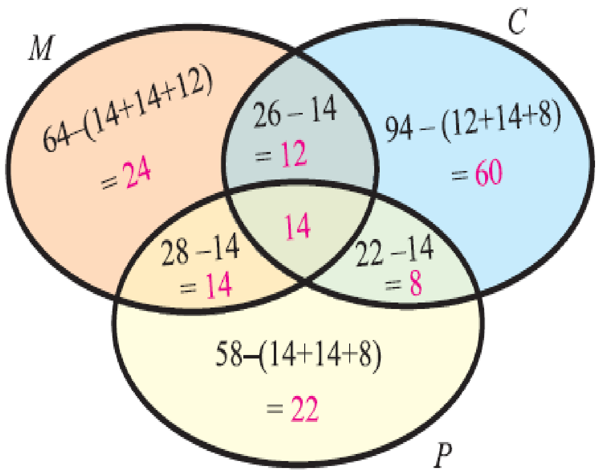
Step 6 : Total no. of students who had taken only one course

= 24 + 60 + 22

= 106

Hence, the total number of students who had taken only one course is 106

Alternative Method (Using venn diagram)



Step 1 : Venn diagram related to the information given in the question:

Step 2 : From the venn diagram above, we have

No. of students who had taken only math = 24

No. of students who had taken only chemistry = 60

No. of students who had taken only physics = 22

Step 3 : Total no. of students who had taken only one course

= 24 + 60 + 22

= 106

Hence, the total number of students who had taken only one course is 106

6. In a group of students, 65 play foot ball, 45 play hockey, 42 play cricket, 20 play foot ball and hockey, 25 play foot ball and cricket, 15 play hockey and cricket and 8 play all the three games. Find the total number of students in the group. (Assume that each student in the group plays at least one game.)

Solution :

Step 1 :Let F, H and C represent the set of students who play foot ball, hockey and cricket respectively.

Step 2 :From the given information, we have

$n(F) = 65$, $n(H) = 45$, $n(C) = 42$,

$n(F \cap H) = 20$, $n(F \cap C) = 25$, $n(H \cap C) = 15$ $n(F \cap H \cap C) = 8$

Step 3 :Total number of students in the group = $n(F \cup H \cup C)$

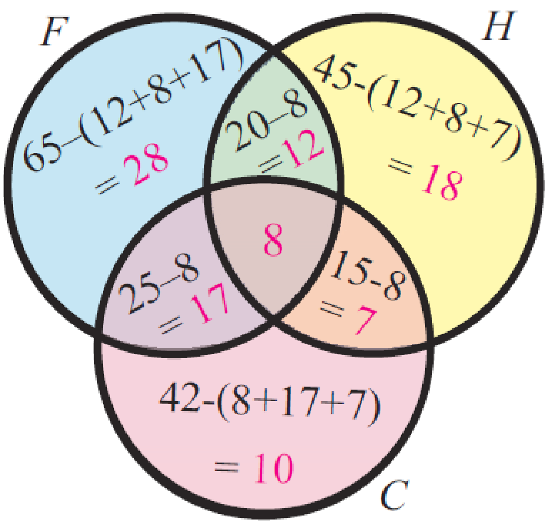
$= n(F) + n(H) + n(C) - n(F \cap H) - n(F \cap C) - n(H \cap C) + n(F \cap H \cap C)$

$= 65 + 45 + 42 - 20 - 25 - 15 + 8$

$= 100$

Hence, the total number of students in the group is 100

Alternative Method (Using venn diagram)



Step 1 :Venn diagram related to the information given in the question:

Step 2 :Total number of students in the group

$= 28 + 12 + 18 + 7 + 10 + 17 + 8 = 100$

Hence, the total number of students in the group is 100

7. There is a group of 80 persons who can drive scooter or car or both. Out of these, 35 can drive scooter and 60 can drive car. Find how many can drive both scooter and car? How many can drive scooter only? How many can drive car only?

Solution:

Let S = {Persons who drive scooter}

C = {Persons who drive car}

Given, $n(S \cup C) = 80$ $n(S) = 35$ $n(C) = 60$

Therefore, $n(S \cup C) = n(S) + n(C) - n(S \cap C)$

$80 = 35 + 60 - n(S \cap C)$

$80 = 95 - n(S \cap C)$

Therefore, $n(S \cap C) = 95 - 80 = 15$

Therefore, 15 persons drive both scooter and car.

Therefore, the number of persons who drive a scooter only = $n(S) - n(S \cap C)$

$= 35 - 15 = 20$

Also, the number of persons who drive car only = $n(C) - n(S \cap C)$

$= 60 - 15 \quad = 45$

8. It was found that out of 45 girls, 10 joined singing but not dancing and 24 joined singing. How many joined dancing but not singing? How many joined both?

Solution:

Let S = {Girls who joined singing}

D = {Girls who joined dancing}

Number of girls who joined dancing but not singing = Total number of girls - Number of girls who joined singing

$= 45 - 24$

$= 21$

Now, $n(S - D) = 10 \qquad n(S) = 24$

Therefore, $n(S - D) = n(S) - n(S \cap D)$

$\Rightarrow n(S \cap D) = n(S) - n(S - D)$

$= 24 - 10$

$= 14$

Therefore, number of girls who joined both singing and dancing is 14

9. If P and Q are two sets such that $P \cup Q$ has 40 elements, P has 22 elements and Q has 28 elements, how many elements does $P \cap Q$ have?

Solution:

Given $n(P \cup Q) = 40$, $n(P) = 18$, $n(Q) = 22$

We know that $n(P \text{ U } Q) = n(P) + n(Q) - n(P \cap Q)$

So, $40 = 22 + 28 - n(P \cap Q)$

$40 = 50 - n(P \cap Q)$

Therefore, $n(P \cap Q) = 50 - 40 \ = 10$

10. How many integers from 1 to 100 are multiples of 2 or 3?

Solution:

Let A be the set of integers from 1 to 100 that are multiples of 2, then $|A| = 50$.

Let B be the set of integers from 1 to 100 that are multiples of 3, then $|B| = 33$.

Now, $A \cap B$ is the set of integers from 1 to 100 that are multiples of both 2 and 3, and hence are multiples of 6, implying $|A \cap B| = 16$.

Hence, by PIE,

$$|A \cup B| = |A| + |B| - |A \cap B| = 50 + 33 - 16 = 67.$$

11. If A = {1, 3, 5}, B = {3, 5, 6} and C = {1, 3, 7}

(i) Verify that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) Verify $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

L.H.S. = $A \cup (B \cap C)$

$B \cap C = \{3\}$

$A \cup (B \cap C) = \{1, 3, 5\} \cup \{3\} = \{1, 3, 5\}$ ……………….. (1)

R.H.S. = $(A \cup B) \cap (A \cup C)$

$A \cup B = \{1, 3, 5, 6\}$

$A \cup C = \{1, 3, 5, 7\}$

$(A \cup B) \cap (A \cup C) = \{1, 3, 5, 6\} \cap \{1, 3, 5, 7\} = \{1, 3, 5\}$ ……………….. (2)

From (1) and (2), we conclude that;

$$A \cup (B \cap C) = A \cup B \cap (A \cup C)$$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

L.H.S. = $A \cap (B \cup C)$

$B \cup C = \{1, 3, 5, 6, 7\}$

$A \cap (B \cup C) = \{1, 3, 5\} \cap \{1, 3, 5, 6, 7\} = \{1, 3, 5\}$ ……………….. (1)

R.H.S. = $(A \cap B) \cup (A \cap C)$

$A \cap B = \{3, 5\}$

$A \cap C = \{1, 3\}$

$(A \cap B) \cup (A \cap C) = \{3, 5\} \cup \{1, 3\} = \{1, 3, 5\}$ ……………….. (2)

Fro (1) and (2), we conclude that;

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

12. Prove that **(A U B)' = A' ∩ B'** (De Morgan's law of union)

Solution:

Let P = (A U B)' and Q = A' ∩ B'

Let x be an arbitrary element of P then $x \in P \Rightarrow x \in (A \cup B)'$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in A' \cap B'$

$\Rightarrow x \in Q$

Therefore, $P \subset Q$ …………….. (i)

Again, let y be an arbitrary element of Q then $y \in Q \Rightarrow y \in A' \cap B'$

$\Rightarrow y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin (A \cup B)$

$\Rightarrow y \in (A \cup B)'$

$\Rightarrow y \in P$

Therefore, $Q \subset P$ …………….. (ii)

Now combine (i) and (ii) we get; $P = Q$ i.e. $(A \cup B)' = A' \cap B'$

13. Prove that $\mathbf{(A \cap B)' = A' \cup B'}$ (De Morgan's law of intersection)

Solution:

Let $M = (A \cap B)'$ and $N = A' \cup B'$

Let x be an arbitrary element of M then $x \in M \Rightarrow x \in (A \cap B)'$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in A' \cup B'$

$\Rightarrow x \in N$

Therefore, $M \subset N$ …………….. (i)

Again, let y be an arbitrary element of N then $y \in N \Rightarrow y \in A' \cup B'$

$\Rightarrow y \in A'$ or $y \in B'$

$\Rightarrow y \notin A$ or $y \notin B$

$\Rightarrow y \notin (A \cap B)$

$\Rightarrow y \in (A \cap B)'$

$\Rightarrow y \in M$

Therefore, $N \subset M$ …………….. (ii)

Now combine (i) and (ii) we get; $M = N$ i.e. $(A \cap B)' = A' \cup B'$

14. Let U = {1, 2, 3, 4, 5, 6, 7, 8}, P = {4, 5, 6} and Q = {5, 6, 8}. Show that (P ∪ Q)' = P' ∩ Q'.

Solution: We know, U = {1, 2, 3, 4, 5, 6, 7, 8}

P = {4, 5, 6} & Q = {5, 6, 8}

P ∪ Q = {4, 5, 6} ∪ {5, 6, 8}

= {4, 5, 6, 8}

Therefore, (P ∪ Q)' = {1, 2, 3, 7} ……………….. (i)

Now P = {4, 5, 6} so, P' = {1, 2, 3, 7, 8}

and Q = {5, 6, 8} so, Q' = {1, 2, 3, 4, 7}

P' ∩ Q' = {1, 2, 3, 7, 8} ∩ {1, 2, 3, 4, 7}

Therefore, P' ∩ Q' = {1, 2, 3, 7} ……………….. (ii)

Combining (i)and (ii) we get;

(P ∪ Q)' = P' ∩ Q'. ……………………………………………Hence Proved

- **Bounded Sets**
  - **Definition 1:** A set S ⊂ R of real numbers is bounded from above if there exists a real number M ∈ R, called an upper bound of S, such that x ≤ M for every x ∈ S.Similarly, S is bounded from below if there exists M ∈ R, called a lower bound of S, such that x ≥ M for every x ∈ S.
  - A set is bounded if it is bounded both from above and below.
  - The supremum of a set is its least upper bound and the infimum is its greatestupper bound.
  - **Definition 2:** Suppose that S ⊂ R is a set of real numbers. If M ∈ R is an upper bound of S such that M ≤ M′ for every upper bound M′of S, then M is called the supremum of S, denoted M= sup S. If M ∈ R is a lower bound of S such that M ≥ M′ for every lower bound M′of S, then M is called the infimum of S, denoted M= inf A.
  - If S is not bounded from above, then we write sup S=∞, and if S is not bounded from below, we write inf S= − ∞.
  - If S=∅ is the empty set, then every real number is both an upper and a lower bound of S, and we write sup∅= − ∞ ,inf ∅ = ∞.

    We will only say the supremum or infimum of a set exists if it is a finite real number.
  - Remarks:
    - ✓ S is bounded iff, S ⊂ [l,u] for some intervel [l,u] of finite length.
    - ✓ S is bounded, iff, there is a positive interger K such that |x| < K for all x ∈ S. Such a number K is called a bound of the set S.

- ✓ Empty set is bounded
- ✓ An upper bound, a lower bound and a bound of a set are not unique.

- Example:
  - ✓ Consider the finite set B = {2,12,0,5,-7,-2} here 12 is upper bounded and -7 is lower bounded. Hence B is bounded.
  - ✓ The set N of natural number is bounded below but not bounded above.
  - ✓ The interval [0,1] is bounded.
  - ✓ Consider set S= {1, 1/2,1/3,1/4……..}. This set consist all numbers of the form 1/n where n ∈ N.We observe that all the number in set S are less than equal to 1 and also observe that no number in set S is less then 0.Thus we say 1 is uppper and 0 is lower bounded respectively for set S.

❖ Unbounded Set

- A set S is unbounded if either it is not bounded above and/or bounded below.
- We know that number u is upper bound of S if the relation $x \leq u$ holds for all $x \in S$. Hence u will not be upper bound of S if there exist some member of S say $y \in S$ such that $y > u$.
- Remark: If a set is bounded above, it has infinitely many upper bounds and similarly if it is bounded below, it has infinitely many lower bounds.
- Example:
  - ✓ Consider sets C={4,6,8,10,……} and D= {0,-1,-2,-3,…..}.
  - ✓ Each element of C is greater than or equal 4. Hence 4 is lower bound of C and thus C is bounded below. From the nature of the element of C, we note that for any number u, however large, there are always elements of C greater than u. Therefore, u cannot be upper bound of C. Thus C has no upper bound.
  - ✓ Similarly, it can be seen that the set D is not bounded below although it is bounded above.Hence both the sets C and D are unbounded sets.

❖ Cantor Diagonalization Argument

- A set S is called Countably infinite if there is a bijection between S and N. That is, you can label the elements of S 1, 2, . . . so that each positive integer is used exactly once as a label.
- Why "Countably infinite"? Such a set is countable because you can count it (via the labeling just mentioned).

- Unlike a finite set, you never stop counting. But at least the elements can be put in correspondence with N.
- On the other hand, **not all infinite sets are countably infinite. In fact, there are infinitely many sizes of infinite sets.**
- Georg Cantor proved this astonishing fact in 1895 by showing that the the set of real numbers is not countable. That is, it is impossible to construct a bijection between N and R. In fact, it's impossible to construct a bijection between N and the interval [0, 1] (whose cardinality is the same as that of R).

Proof :

Suppose that f : N → [0, 1] is any function. Make a table of values of f, where the 1^{st} row contains the decimal expansion of $f(1)$, the 2^{nd} row contains the decimal expansion of $f(2)$, . . . the n^{th} row contains the decimal expansion of $f(n)$, . . . Perhaps $f(1) = \pi/10, f(2) = 37/99, f(3) = 1/7, f(4) = \sqrt{2}/2, f(5) = 3/8$.

n	f(n)
1	0 . 3 1 4 1 5 9 2 6 5 3 . . .
2	0 . 3 7 3 7 3 7 3 7 3 7 . . .
3	0 . 1 4 2 8 5 7 1 4 2 8 . . .
4	0 . 7 0 7 1 0 6 7 8 1 1 . . .
5	0 . 3 7 5 0 0 0 0 0 0 0 . . .
	…………
	…………

Can f possibly be onto? That is, can every number in [0, 1] appear somewhere in the table?

In fact, the answer is no — there are lots and lots of numbers that can't possibly appear! For example, let's highlight the digits in the main diagonal of the table.

n	f(n)
1	0 . **3** 1 4 1 5 9 2 6 5 3 . . .
2	0 . 3 **7** 3 7 3 7 3 7 3 7 . . .
3	0 . 1 4 **2** 8 5 7 1 4 2 8 . . .
4	0 . 7 0 7 **1** 0 6 7 8 1 1 . . .
5	0 . 3 7 5 0 **0** 0 0 0 0 0 . . .
	…………
	…………

The highlighed digits are **0.37210 . . . .** Suppose that we add 1 to each of these digits, to get the number **0.48321 . . . .**

Now, this number can't be in the table. Why not? Because

- ✓ it differs from $f\,(1)$ in its first digit;
- ✓ it differs from $f\,(2)$ in its second digit;
- ✓ . . .
- ✓ it differs from $f\,(\text{n})$ in its n^{th} digit;
- ✓ . . .

Suppose that we subtract 1 to each of these digits, to get the number **0.26109 . . . .** Consequently, we can say that the list above is not an exhaustive listing of the set pf all real number 0 and 1, a contradiction to our assumption. Hence the set of real number between 0 and 1 is uncountable.

❖ Countably Infinite and Uncountably Infinite Sets

- ➢ A set is **countably infinite** if its elements can be put in one-to-one correspondence with the set of natural numbers. In other words, one can count off all elements in the set in such a way that, even though the counting will take forever, you will get to any particular element in a finite amount of time.
- ➢ Example: The integers Z form a countable set.
- ➢ A set is **uncountable** if it contains so many elements that they cannot be put in one-to-one correspondence with the set of natural numbers. In other words, there is no way that one can count off all elements in the set in such a way that, even though the counting will take forever, you will get to any particular element in a finite amount of time. **OR**
- ➢ In mathematics, an **uncountable set** (or uncountably infinite set) is an infinite set that contains too many elements to be countable. The uncountability of a set is closely related to its cardinal number: a set is uncountable if its cardinal number is larger than that of the set of all natural numbers.
- ➢ Example of an uncountable set is the set R of all real numbers;

❖ Multiset or bags

- ➢ A generalization of the concept of set in which elements may appear multiple times: an unordered sequence of elements. **OR**
- ➢ A multiset (**mset**, for short) is an unordered collection of objects (called the elements) in which, unlike a standard (Cantorian) set, elements are allowed to repeat. **OR**
- ➢ In other words, an mset is a set to which elements may belong more than once, and hence it is a non-Cantorian set.

- The number of copies of an element appearing in an mset is called its multiplicity.
- The number of distinct elements in an mset M (which need not be Finite) and their multiplicities jointly determine its cardinality, denoted by C(M).
- In other words, the cardinality of an mset is the sum of multiplicities of all its elements.
- An mset M is called Finite if the number of distinct elements in M and their multiplicities are both Finite, it is infinite otherwise.
- Example: The multisets {a,a,b}, {a,b,a} and {b,a,a} are the same but not equal to either {a,b,b} or to {a,b}.
- Two important Characteristics is of Msets:
  - ✓ There may be repeated occurrences of elements.
  - ✓ There is no particular order or arrangement of the elements.
- In fact we can characterize a multiset as a pair of (A, μ),where A is generic set and μ is the multiplicity function defined as

  $$\mu: A \rightarrow \{1, 2, 3, \ldots\}$$

  so that μ(a) = k, where k is number of times the element a occur in the mset.
- For Example: if [a, b, c, c, a, c] is the mset, μ(a)=2, μ(b)=1, and μ(c)=3.

1. Equality of Multiset

- If the number of occurrences of each element is the same in both the msets, then the msets are equal.
- Example: [a,b,a,a] = [a,a,b,a] and [a,b,a] ≠ [a,b]

2. Multisubset or Msubset

- A multiset A is said to be a multisubset of B if multiplicity of each element in A is less or equal to its multiplicity in B.
- Example: [1,2,2,3] ⊆ [1,1,1,2,2,3]

3. Union and Intersection of Msets

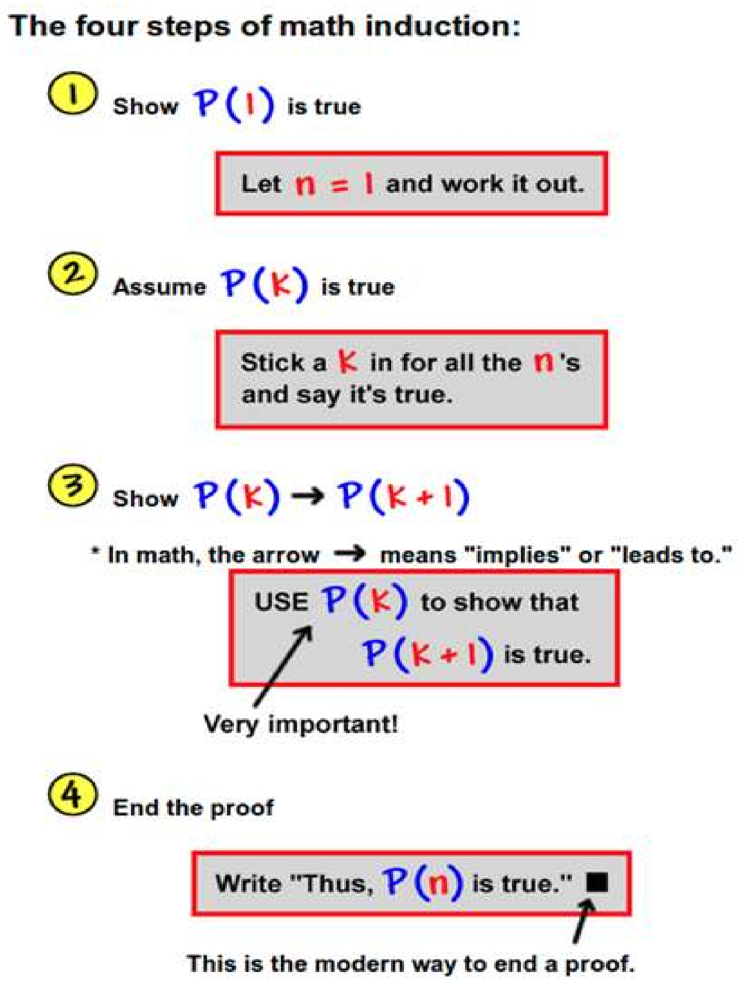
- Let A and B be Msets, and *m* and *n* be the number of times *x* occurs in A and B respectively. Put the larger of *m* and *n* occurrences of *x* in A U B. Put the smaller of *m* and *n* occurrences of *x* in A ∩ B.
- For Example 1: A= {2, 2, 3} and B= {2, 3, 3, 4}

  A U B = [2, 2, 3] U [2, 3, 3, 4] = [2, 2, 3, 3, 4]

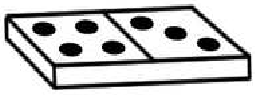
  A ∩ B = [2, 2, 3] ∩ [2, 3, 3, 4] = [2,3].

❖ **Mathematical Induction**

- Mathematical Induction is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural number.
- The technique involves two steps to prove a statement, as stated below –
  - ✓ **Step 1 (Base step)**: It proves that a statement is true for the initial value. (i. e. $n=n_0$)
  - ✓ **Step 2 (Inductive step)** – It proves that if the statement is true for the n^{th} iteration (or number n), then it is also true for $(n+1)^{th}$ iteration (or number n+1).
  - ✓ (i.e. Statement is true for n=k+1, assuming that it is true for n=k, $(k \geq n_0)$)
- **How to Do It:**
- **Step 1**– Consider an initial value for which the statement is true. It is to be shown that the statement is true for n = initial value.
- **Step 2**– Assume the statement is true for any value of n = k. Then prove the statement is true for n = k+1. We actually break n = k+1 into two parts, one part is n = k (which is already proved) and try to prove the other part.
- Above two step are divided into four Step of math induction which is as follows:

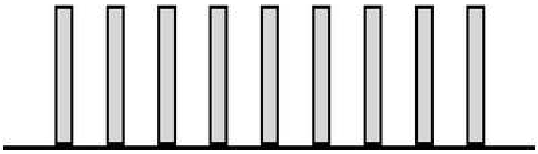


Let's look at some dominoes...



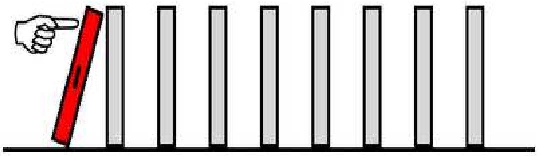
Did you ever stack them so you could knock them all down? It's actually pretty fun and, if you've never done it, I highly recommend that you do.

Let's line up a row of dominoes...



There are four main parts to math induction...

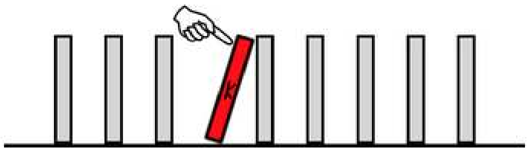
(1) Can we knock down the first domino?



Yes!

Show $P(1)$ is true.

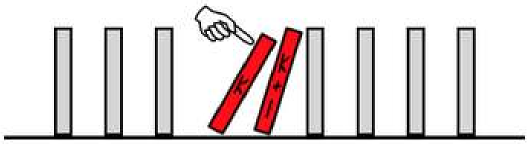
(2) Can we knock down a random domino somewhere in the middle?
Let's call it the kth domino.



Yes!

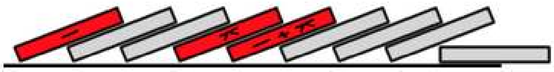
Assume $P(k)$ is true.

(3) (This one is the big deal.)
If we knock down that kth domino, will the next domino get knocked down too?



Show $P(k) \rightarrow P(k+1)$.

(4) If we do all of the above, will all the dominoes fall?



YES!

Thus, $P(n)$ is true.

- **Example 1:**

Prove

$$P(n):\ 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

(1) Show $P(1)$ is true:

Stick a 1 in for all the n's and show it works.

$$P(1):\ 1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

left side ↗   right side ↗

So, $P(1)$ is true.

② Assume $P(k)$ is true:

Stick a k in for all the n's and say it's true.

$$P(k):\ 1+2+3+\ldots+k = \frac{k(k+1)}{2}$$

is true.

③ Show $P(k) \rightarrow P(k+1)$

Use $P(k)$ to show that $P(k+1)$ is true.

Write out your goal by sticking $(k+1)$ in for all the n's... Leave the k on the left side.

GOAL: $P(k+1)$:

$$1+2+3+\ldots+k+(k+1) = \frac{(k+1)[(k+1)+1]}{2}$$

Start with the left side of the $P(k+1)$ and slowly turn it into how the right side looks.

$$P(k+1):\ \underbrace{1+2+3+\ldots+k}_{\text{left side of } P(k)}+(k+1)$$

$$= \underbrace{\frac{k(k+1)}{2}}_{\text{right side of } P(k)} + (k+1)$$

Remember to use $P(k)$!

Let your goal guide you!

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1) + 2(k+1)}{2}$$

need a 2 on the bottom

make one fraction

Notice that there is a $(k+1)$ in the front of your goal, so factor that out.

$$= \frac{(k+1) + (k+2)}{2} = \frac{(k+1)[(k+1)+1]}{2}$$

just a little rewrite here

So, $P(k) \rightarrow P(k+1)$

(4) Thus, $P(n)$ is true. ■

❖ **Example 2:**

Prove

$$P(n):\ 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(1) Show $P(1)$ is true:

Stick a 1 in for all the n's and show it works.

$$P(1):\ 1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = \frac{6}{6} = 1$$

So, $P(1)$ is true.

② Assume $P(k)$ is true:

Stick a k in for all the n's and say it's true.

$$P(k):\ 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

is true

③ Show $P(k) \rightarrow P(k+1)$

Use $P(k)$ to show that $P(k+1)$ is true.

Write out your goal by sticking $(k+1)$ in for all the n's... Leave the k on the left side.

GOAL: $P(k+1)$:

$$P(k+1):\ 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

Start with the left side of the $P(k+1)$ and slowly turn it into the right side... Don't confuse the monkey!

$$P(k+1):\ \underbrace{1^2 + 2^2 + 3^2 + \ldots + k^2}_{P(k)} + (k+1)^2$$

Use $P(k)$!

$$= \underbrace{\frac{k(k+1)(2k+1)}{6}}_{P(k)} + (k+1)^2$$

need that 6 and one big fraction

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

Look at the end goal... What's in front? Factor it out!

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$ ← Simplify this end part...

$$= \frac{(k+1)[2k^2 + k + 6k + 6]}{6}$$

factor this... it MUST work!

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

You can look at the goal to cheat!

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$ ← now, make these last parts look like the goal

$$= \frac{(k+1)[(k+1)+1][2k+2+1]}{6}$$

$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

So, $P(k) \rightarrow P(k+1)$

(4) Thus, $P(n)$ is true. ■

❖ **Example 3:**

Prove $P(n): \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

(1) Show $P(1)$ is true:

$$P(1): \left(\frac{a}{b}\right)^1 = \frac{a}{b} = \frac{a^1}{b^1}$$

↑

A picky but important note is that you can't skip this middle step. If you do, then you'd be using the rule you are trying to prove... which is circular reasoning.

So, $P(1)$ is true.

(2) Assume $P(k)$ is true:

$P(k): \left(\frac{a}{b}\right)^k = \frac{a^k}{b^k}$ is true

(3) Show $P(k) \rightarrow P(k+1)$

Use $P(k)$ AND $P(1)$ to show that $P(k+1)$ is true.

GOAL: $P(k+1): \left(\frac{a}{b}\right)^{k+1} = \frac{a^{k+1}}{b^{k+1}}$

Start with the left side... Use $P(k)$ and $P(1)$ show every little step... Don't confuse the monkey... End up with the right side.

$$P(k+1): \left(\frac{a}{b}\right)^{k+1} = \underset{\uparrow\, P(k)}{\left(\frac{a}{b}\right)^{k}} \underset{\uparrow\, P(1)}{\left(\frac{a}{b}\right)^{1}} = \underset{\uparrow\, P(k)}{\frac{a^k}{b^k}} \cdot \underset{\uparrow\, P(1)}{\frac{a^1}{b^1}}$$

$$= \frac{a^k a^1}{b^k b^1} = \frac{a^{k+1}}{b^{k+1}}$$

So, $P(k) \rightarrow P(k+1)$

(4) Thus, $P(n)$ is true. ■

❖ **Example 4:**

$3^n - 1$ is a multiple of 2 for n = 1, 2, ...

Solution

Step 1 – For $n = 1, 3^1 - 1 = 3 - 1 = 2$ which is a multiple of 2

Step 2 – Let us assume $3^n - 1$ is true for $n = k$, Hence, $3^k - 1$ is true (It is an assumption)

We have to prove that $3^{k+1} - 1$ is also a multiple of 2

$3^{k+1} - 1 = 3 \times 3^k - 1 = (2 \times 3^k) + (3^k - 1)$

The first part $(2 \times 3k)$ is certain to be a multiple of 2 and the second part $(3k - 1)$ is also true as our previous assumption.

Hence, $3^{k+1} - 1$ is a multiple of 2.

So, it is proved that $3^n - 1$ is a multiple of 2.

- **Example 5:**

Prove that $(ab)^n = a^n b^n$ is true for every natural number n

Solution

Step 1 – For $n = 1, (ab)^1 = a^1 b^1 = ab$, Hence, step 1 is satisfied.

Step 2 – Let us assume the statement is true for $n = k$, Hence, $(ab)^k = a^k b^k$ is true (It is an assumption).

We have to prove that $(ab)^{k+1} = a^{k+1} b^{k+1}$ also hold

Given, $(ab)^k = a^k b^k$

Or, $(ab)^k (ab) = (a^k b^k)(ab)$ [Multiplying both side by 'ab']

Or, $(ab)^{k+1} = (aa^k)(bb^k)$

Or, $(ab)^{k+1} = (a^{k+1} b^{k+1})$

Hence, step 2 is proved.

So, $(ab)^n = a^n b^n$ is true for every natural number n.

- **Example 6:**

$1 + 3 + 5 + \ldots + (2n - 1) = n^2$ for $n = 1, 2, \ldots$

Solution

Step 1 – For $n = 1, 1 = 1^2$, Hence, step 1 is satisfied.

Step 2 – Let us assume the statement is true for $n = k$.

Hence, $1 + 3 + 5 + \cdots + (2k - 1) = k^2$ is true (It is an assumption)

We have to prove that $1 + 3 + 5 + \ldots + (2(k + 1) - 1) = (k + 1)^2$ also holds

$$1 + 3 + 5 + \cdots + (2(k + 1) - 1)$$

$$= 1 + 3 + 5 + \cdots + (2k + 2 - 1)$$

$$= 1 + 3 + 5 + \cdots + (2k + 1)$$

$$= 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1)$$

$$= k^2 + (2k + 1)$$

$$= (k + 1)^2$$

So, $1 + 3 + 5 + \cdots + (2(k + 1) - 1) = (k + 1)^2$ hold which satisfies the step 2.

Hence, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ is proved.

❖ PROPOSITIONAL LOGIC

❖ Introduction

- A statement can be defined as a declarative sentence, or part of a sentence, that is capable of having a truth-value, such as being true or false.
- A propositional consists of propositional variables and connectives. The propositional variables are denoted by capital letters (A, B, etc) and connectives connect the propositional variables.
- All the following declarative sentences are propositions:
  1. The sun rises in the East and sets in the West.
  2. Narendra Modi is the 14^{th} Prime Minister of India.
  3. Mumbai is the capital of India.
  4. 1 + 1 = 2.
  5. 2 + 2 = 3.
- Propositions 1,2 and 4 are true, whereas 3 and 5 are false.
- Now Consider the following sentences:
  1. What time is it?
  2. Read this carefully
  3. x + 1 = 2.
  4. x + y = z.
- Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables.
- Sometimes, a statement can contain one or more other statements as parts.
- When two statements are joined together with "and", the complex statement formed by them is true if and only if both the component statements are true. Because of this, an argument of the following form is logically valid:
  - ✓ Paris is the capital of France and Paris has a population of over two million.
- In propositional logic generally we use five connectives which are −
  - ✓ Negation/ NOT (¬)
  - ✓ OR (∨) (Disjunction)
  - ✓ AND (∧) (Conjunction)
  - ✓ Exclusive OR (⊕)
  - ✓ Implication / if-then (→) (Conditional or Implication)
  - ✓ If and only if (⇔) (Bi-conditional or Double Implication)

A	B	¬A	A ∨ B	A ∧ B	A ⊕ B	A → B	A ⇔ B
T	T	F	T	T	F	T	T
T	F	F	T	F	T	F	F
F	T	T	T	F	T	T	F
F	F	T	F	F	F	T	T

❖ **Tautologies:**

➢ A Tautology is a formula which is always true for every value of its propositional variables.

➢ Example: Prove $[(A \rightarrow B) \wedge A] \rightarrow B$ is a tautology

➢ The truth table is as follows:

A	B	A → B	(A → B) ∧ A	[(A → B) ∧ A] → B
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

❖ **Contradictions:**

➢ A Contradiction is a formula which is always false for every value of its propositional variables.

➢ Example: Prove $(A \vee B) \wedge [(\neg A) \wedge (\neg B)]$ is a contradiction

A	B	A ∨ B	¬ A	¬ B	(¬ A) ∧ (¬ B)	(A ∨ B) ∧ [(¬ A) ∧ (¬ B)]
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

❖ **Contingency:**

➢ A Contingency is a formula which has both some true and some false values for every value of its propositional variables.

➢ Example: Prove (A∨B) ∧ (¬A) a contingency

➢ The truth table is as follows:

A	B	A ∨ B	¬ A	(A ∨ B) ∧ (¬ A)
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

❖ **Propositional Equivalences:**

➢ Two statements X and Y are logically equivalent if any of the following two conditions hold:

- ✓ The truth tables of each statement have the same truth values.
- ✓ The bi-conditional statement $X \Leftrightarrow Y$ is a tautology.

- Example: Prove ¬ (A∨B) and [(¬A)∧(¬B)] are equivalent
- **Testing by 1st method (Matching truth table):**

A	B	A ∨ B	¬ (A ∨ B)	¬ A	¬ B	[(¬ A) ∧ (¬ B)]
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

- Here, we can see the truth values of ¬ (A∨B) and [(¬A) ∧ (¬B)] are same, hence the statements are equivalent.
- **Testing by 2nd method (Bi-conditionality)**

A	B	¬ (A ∨ B)	[(¬ A) ∧ (¬ B)]	[¬ (A ∨ B)] ⇔ [(¬ A) ∧ (¬ B)]
T	T	F	F	T
T	F	F	F	T
F	T	F	F	T
F	F	T	T	T

- ✓ As $[\neg (A \vee B)] \Leftrightarrow [(\neg A) \wedge (\neg B)]$ is a tautology, the statements are equivalent.

❖ **Inverse, Converse, and Contra-positive**

- Implication / if-then (→) is also called a conditional statement. It has two parts −
- ✓ Hypothesis, P
- ✓ Conclusion, Q
- As mentioned earlier, it is denoted as P→Q.
- **Example of Conditional Statement** − "If you do your homework, you will not be punished." Here, "you do your homework" is the hypothesis, P, and "you will not be punished" is the conclusion, Q.

❖ **Inverse**

- An inverse of the conditional statement is the negation of both the hypothesis and the conclusion. If the statement is "If P, then Q", the inverse will be "If not P, then not Q".
  Thus the inverse of P→ Q is ¬ P→ ¬ Q.
- **Example** − The inverse of "If you do your homework, you will not be punished" is "If you do not do your homework, you will be punished."

❖ **Converse**

- The converse of the conditional statement is computed by interchanging the hypothesis and the conclusion. If the statement is "If P, then Q", the converse will be "If Q, then P".
  The converse of P → Q is Q → P.

- **Example** − The converse of "If you do your homework, you will not be punished" is “If you will not be punished, you do your homework”.

- **Contra-positive**
  - The contra-positive of the conditional is computed by interchanging the hypothesis and the conclusion of the inverse statement. If the statement is “If P, then Q”, the contra-positive will be “If not Q, then not P”.
    The contra-positive of $P \rightarrow Q$ is $\neg Q \rightarrow \neg P$.
  - **Example** − The Contra-positive of "If you do your homework, you will not be punished” is “If you are punished, you did not do your homework”.

- **Logical Equivalences**

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan’s laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

- **Example:**

1. Prove that $P \vee \neg P$ is a Tautology

P	**¬P**	**P ∨ ¬ P**
T	F	T
F	T	T

2. Prove that $P \wedge \neg P$ is a Contradiction

P	¬P	P ∧ ¬ P
T	F	F
F	T	F

3. Construct the truth table for (P → Q) ∧ (¬P ⇔Q)

P	Q	¬P	P → Q	¬P ⇔Q	(P → Q) ∧ (¬P ⇔Q)
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

4. Determine whether each of the following is a Tautology, a Contradiction or Neither:

a) [P ∧ (P → Q)] → Q

b) (P → Q) ↔ (¬ Q → ¬ P)

c) (¬ P ∧ Q) ∧ (P ∨ ¬Q)

d) (P → ¬ Q) ∨ (¬ R → P)

e) (P → Q) ∧ (¬ P ∨ Q)

f) (P → Q) → (P ∧ Q)

Solution:

a) [P ∧ (P → Q)] → Q

P	Q	P → Q	P ∧ (P → Q)]	[P ∧ (P → Q)] → Q
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

b) (P → Q) ↔ (¬ Q → ¬ P)

P	Q	P → Q	¬ Q	¬ P	¬ Q → ¬ P	(P → Q) ↔ (¬ Q → ¬ P)
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

c) (¬ P ∧ Q) ∧ (P ∨ ¬Q)

P	Q	¬ Q	¬ P	(¬ P ∧ Q)	(P ∨ ¬Q)	(¬ P ∧ Q) ∧ (P ∨ ¬Q)
T	T	F	F	F	T	F
T	F	T	F	F	T	F
F	T	F	T	T	F	F
F	F	T	T	F	T	F

d) (P → ¬ Q) ∨ (¬ R → P)

P	Q	R	¬ Q	P → ¬ Q	¬ R	¬ R → P	(P → ¬ Q) ∨ (¬ R → P)
T	T	T	F	F	F	T	T

T	T	F	F	F	T	T	T
T	F	T	T	T	F	T	T
T	F	F	T	T	T	T	T
F	T	T	F	T	F	T	T
F	T	F	F	T	T	F	T
F	F	T	T	T	F	T	T
F	F	F	T	T	T	F	T

e) (P → Q) ∧ (¬ P ∨ Q)

P	Q	P → Q	¬ P	¬ P ∨ Q	(P → Q) ∧ (¬ P ∨ Q)
T	T	T	F	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	T	T	T

f) (P → Q) → (P ∧ Q)

P	Q	P → Q	P ∧ Q	(P → Q) → (P ∧ Q)
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

5. Given the tree propositions P, Q and R, construct truth tables for:
   a) (P ∧ Q) → ¬ R
   b) P ∧ (¬ Q ∨ R)
   c) P → (¬ Q ∨ ¬ R)
6. What are the contrapositive, the converse, & the inverse of the conditional statement? "The home team wins whenever it is raining?"

   Solution: Because "q whenever p" is one of the ways to express the conditional statement p→q, the original statement can be rewritten as:

   "If it is raining, then the home team wins."

   Consequently, the contrapositive of this conditional statement is "If the home team does not win, then it is not raining."

   The converse is "If the home team wins, then it is raining."

   The inverse is "If it is not raining, then the home team does not win."

   Only the contrapositive is equivalent to the original statement.
7. If P→Q is false, determine the value of (¬ (P∧Q)) →Q

P	Q	P→Q	(P∧Q)	¬ (P∧Q)	(¬ (P∧Q)) →Q
T	F	F	F	T	F

8. If P & Q are false , find truth values of (P ∨ Q) ∧ (¬ P ∨ ¬ Q)

P	Q	¬P	¬Q	P ∨ Q	¬ P ∨ ¬ Q	(P ∨ Q) ∧ (¬ P ∨ ¬ Q)
F	F	T	T	F	T	F

9. If P→Q is true, Can we determine the value ¬ P ∨ (P→Q)

P	Q	¬ P	P → Q	¬ P ∨ (P→Q)
T	T	F	T	T
F	T	T	T	T
F	F	T	T	T

❖ **Applications of Propositional Logic**

- Logic has many important applications to mathematics, computer science, and numerous other disciplines.
- Statements in mathematics and the sciences in natural language often are imprecise or ambiguous.
- To make such statements precise, they can be translated into the language of logic. For example, logic is used in the specification of software and hardware, because these specifications need to be precise before development begins.
- Furthermore, propositional logic and its rules can be used to design computer circuits, to construct computer programs, to verify the correctness of programs, and to build expert systems.
- Logic can be used to analyze and solve many familiar puzzles. Software systems based on the rules of logic have been developed for constructing some, but not all, types of proofs automatically.

❖ **Translating English Sentences**

- There are many reasons to translate English sentences into expressions involving propositional variables and logical connectives.
- In particular, English (and every other human language) is often ambiguous.
- Translating sentences into compound statements removes the ambiguity.
- Basic three steps for translation are:
  - ✓ Step 1: Find logical connectives.
  - ✓ Step 2: Break the sentence into elementary propositions.
  - ✓ Step 3: Rewrite the sentence in propositional logic.

❖ **Example:**

1. You can have free coffee if you are senior citizen and it is a Tuesday

   Solution:

   Step 1: Find logical connectives.

   You can have free coffee **if** you are senior citizen **and** it is a Tuesday

   Step 2: Break the sentence into elementary propositions.

A: You can have free coffee

B: You are senior citizen

C: It is a Tuesday

Step 3: Rewrite the sentence in propositional logic.

$$(B \wedge C) \rightarrow A$$

2. Assume two elementary statements:

   P: you drive over 65 mph; Q: you get a speeding ticket.

   Translate each of these sentences to logic

   - ✓ You do not drive over 65 mph. ($\neg P$).
   - ✓ You drive over 65 mph, but you don't get a speeding ticket. ($P \wedge \neg Q$).
   - ✓ You will get a speeding ticket if you drive over 65 mph. ($P \rightarrow Q$).
   - ✓ If you do not drive over 65 mph then you will not get a speeding ticket. ($\neg P \rightarrow \neg Q$).
   - ✓ Driving over 65 mph is sufficient for getting a speeding ticket. ($P \rightarrow Q$)
   - ✓ You get a speeding ticket, but you do not drive over 65 mph. ($Q \wedge \neg P$).

3. If you are older than 15 or you are with your parents then you can play roll coaster.

   Solution:

   Step 1: Find logical connectives.

   If you are older than 15 **or** you are with your parents then you can play roll coaster.

   Step 2: Break the sentence into elementary propositions.

   A= you are older than 15

   B= you are with your parents

   C= you can play roll coaster

   Step 3: Rewrite the sentence in propositional logic.

   $$(A \vee B) \rightarrow C$$

4. Express the specification "The automated reply cannot be sent when the file system is full" using logical connectives.

   Solution: Let P denote "The automated reply can be sent" and Q denote "The file system is full." $Q \rightarrow \neg P$

5. Translate the following sentence into propositional logic: "You can access the Internet from campus only if you are a computer science major or you are not a freshman."

   Solution: Let A, C, and F represent respectively "You can access the internet from campus," "You are a computer science major," and "You are a freshman."

   $$A \rightarrow (C \vee \neg F)$$

6. Let P and Q be the propositions: "The election is decided" and "the votes have been counted" respectively. Express each of the propositions as English sentences:
   a) $\neg P$
   b) $P \vee Q$
   c) $\neg P \wedge Q$
   d) $Q \rightarrow P$
   e) $\neg P \rightarrow \neg Q$
   f) $P \Leftrightarrow Q$
   g) $\neg Q \vee (\neg P \wedge Q)$

   Solution:
   a) The election is not (yet) decided.
   b) The election is decided or the votes have been counted.
   c) The votes have been counted but the election is not (yet) decided.
   d) If the votes have been counted then the election is decided.
   e) The election is not decided unless the votes have been counted.
   f) The election is decided if and only if the votes been counted.
   g) The votes have not been counted, or they have been counted by the election is not (yet) decided.