Unit II: Relation & Function 07 Hours

- **Relations** and their Properties, n-ary relations and their applications, Representing relations, Closures of relations, Equivalence relations, Partial orderings, Partitions, Hasse diagram, Lattices, Chains and Anti-Chains, Transitive closure and Warshall‘s algorithm.
- **Functions-**Surjective, Injective and Bijective functions, Identity function, Partial function, Invertible function, Constant function, Inverse functions and Compositions of functions, The Pigeonhole Principle.
- **#Exemplar/Case Studies:** Know about the great philosophers-Dirichlet.

❖ Relation-Introduction

- **Relationships between elements of sets are represented using the structure called a relation, which is just a subset of the Cartesian product of the sets.**
- Every day we deal with relationships such as those between a business and its telephone number, an employee and his or her salary, a person and a relative, and so on.
- Relations can be used to solve problems such as determining which pairs of cities are linked by airline flights in a network, finding a viable order for the different phases of a complicated project, or producing a useful way to store information in computer databases.
- A **relation** is any association or link between elements of one set, called the **domain** or (less formally) the set of inputs, and another set, called the **range** or set of outputs.
- A common notion of relation is a type of association that exists between two or more objects.
- **Example 1:** “Is the mother of” is a relation between the set of all females and the set of all people. It consists of all the pairs (person 1, person 2) where person 1 is the mother of person 2.
- Also Consider following Example:
  - ✓ x is the father of y.
  - ✓ The number x is greater than the number y.
- From above example it’s clear that order of object is very important.

❖ Binary Relation

- A relation is an association between two or more things, when it exists between two elements then it is called **Binary relation.**
- **A (binary) relation** R between the sets A & B **(written as R: A ↔ B)** is a subset of the Cartesian product **A × B.i.e. R ⊆ A × B**

- If (x, y) ∈ R, we say x is related to y. We denote this by xRy.
- The set A is called the **domain** of the relation and the set B the **codomain.**
- **Domain =** Set of first elements in the Cartesian product.
- **Range =** Set of second elements in the Cartesian product.
- If there are two sets A and B and Relation from A to B is R(a,b), then domain is defined as the set { a | (a,b) ∈ R for some b in B} and Range is defined as the set {b | (a,b) ∈ R for some a in A}.
- **Example 2:** Let A= {a, b, c} and B= {1, 2, 3}.

  R= {(a, 1), (b, 2), (c, 3)} is an example of a relation from A to B.
- **Example 3:** Let A = {1, 2, 3, 4}. Which ordered pairs are in the relation?
  - ✓ R = {(a, b) | a < b}?
  - ✓ R = {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}. Domain= {1, 2, 3} & Range= {2, 3, 4}

❖ Types of Relation

- The **Empty Relation** between sets A and B, or on E, is the empty set ∅
  - ✓ Example: If set A = {1, 2, 3} then, one of the void relations can be R = {x, y} where, |x – y| = 8. Here R = ∅ **⊆ A × A**

- The **Full Relation** between sets A and B is the set A × B.

- The **Identity Relation** on set A is the set {(x, x) | x ∈ A}
  - ✓ Example: Let A={1,2,3} then I_A={(1,1),(2,2),(3,3)}
  - ✓ Example: Let A = (1, 2, 3, 4, 5) and R = {(1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,4), (3,3), (4,4), (5,5)}. Then I_A = {(1,1),(2,2),(3,3),(4,4),(5,5)}

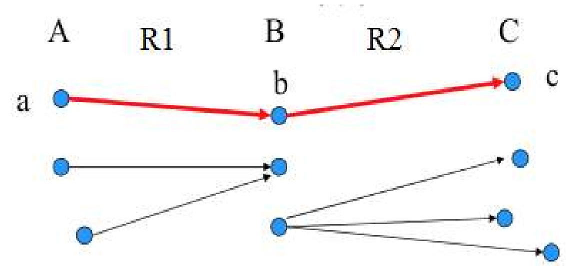
- The Relation R in set A is said to **Universe Relation** if R= A x A
  - ✓ Example: A={a,b,c} then R =A × A={(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)}
  - ✓ Example: Let A = (1, 2, 3, 4, 5) and R = {(1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,4), (3,3), (4,4), (5,5)}.
  - ✓ U = {(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4), (2,5), (3,1), (3,2), (3,3), (3,4), (3,5), (4,1), (4,2), (4,3), (4,4), (4,5), (5,1), (5,2), (5,3), (5,4), (5,5)}

- The **Inverse Relation R' (Converse Relation R^C)** of a relation R is defined as – R' or R^C = {(b, a) | (a, b) ∈ R}. Let R be a relation from set A to set B, then inverse relation R' is from set B to set A.
  - ✓ Example: Let R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 4), (3, 3), (4, 4), (5, 5)}

- ✓ R' ={(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (2, 2), (4, 2), (3, 3), (4, 4), (5, 5)}

- The **Complement of a relation** R is defined as $\hat{R}$={(a, b) | (a,b) ∉ R}. i.e. a $\hat{R}$ b iff a $\not R$ b.
  - ✓ $\hat{R}$ = (A x B) – R
  - ✓ Note: $\hat{R}$ is the universe of discourse is U = A × B; thus the name complement.
  - ✓ Example: Let A = (1, 2, 3, 4, 5) and R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 4), (3, 3), (4, 4), (5, 5)}
  - ✓ $\hat{R}$ = {(2, 1), (2, 3), (2, 5), (3, 1), (3, 2), (3, 4), (3, 5),(4, 1), (4, 2), (4, 3), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4)}

- The **Composite Relation:** Let R1 be a binary relation from a set A to a set B, R2 a binary relation from B to a set C. Then the composite relation from A to C denoted by R1.R2.
  R1. R2 = {(a,c) | aR1b, bR2c ; for a ∈ A, c ∈ C}



  - ✓ Example: A = {1, 2, 3}, B = {a, b, c, d}, C = {x, y, z}.
    R : A ↔ B,   R = {(1, a), (1, b), (2, b), (2, c)}.
    S : B ↔ C,   S = {(a, x), (a, y), (b, y), (d, z)}.
    R.S = {(1, x), (1, y), (2, y)}.

- **Combining Relations:** Since relations from A to B are subsets of **A x B**, two relations from A to B can be combined through set operations.

- **Powers of a Relation**: Let R be a relation on the set A. The powers R^n, n = 1, 2, 3, . . . , are defined recursively by $\mathbf{R^1 = R}$ **and** $\mathbf{R^{n+1} = R^n \circ R}$.
  - ✓ The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on.

- **Example 4:** Let A = {1, 2, 3} & B = {1, 2, 3, 4}. The relations R_1 = {(1, 1), (2, 2), (3, 3)} and R_2 = {(1, 1), (1, 2), (1, 3), (1, 4)} can be combined to obtain
  R1 ∪ R2 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)},
  R1 ∩ R2 = {(1, 1)},
  R1 − R2 = {(2, 2), (3, 3)},

R2 − R1 = {(1, 2), (1, 3), (1, 4)}.

- **Example 5:** Let A be the set of students at your school and B the set of books in the school library. Let R1 and R2 be the relations consisting of all ordered pairs (a, b), where student a is required to read book b in a course, and where student a has read book b, respectively. Describe the ordered pairs in each of these relations.

  a) R1 ∪ R2 b) R1 ∩ R2 c) R1 ⊕ R2 d) R1 − R2 e) R2 − R1

  Solution: the set of pairs (a, b) where

  a) a is required to read b in a course or has read b.
  b) a is required to read b in a course and has read b.
  c) a is required to read b in a course or has read b, but not both; equivalently, the set of pairs (a, b) where a is required to read b in a course but has not done so, or has read b although not required to do so in a course.
  d) a is required to read b in a course but has not done so.
  e) a has read b although not required to do so in a course.

- **Example 6 a):** Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.

  R = {(1,0), (1,2), (3,1),(3,2)} and S = {(0,b), (1,a) ,(2,b)}

  R.S = {(1,b),(3,a),(3,b)}

- **Example 6 b):** Let R be the relation {(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)}, and let S be the relation {(2, 1), (3, 1), (3, 2), (4, 2)}. Find R.S

  Solution: R.S = {(1,1), (1,2), (2,1) (2, 2)}

- **Example 7:** Let A = (1,2,3,4), Let R1 = {(x,y)| x + y = 5} and R2 = {(x,y)| y - x = 1}. Verify $(R1.R2)^C = R2^C. R1^C$.

  Solution: R1 = {(1,4),(2,3),(3,2),(4,1)} & R2 = {(1,2),(2,3),(3,4)}

  (R1.R2) = {(2,4),(3,3),(4,2)}

  $(R1.R2)^C$ = {(4,2),(3,3),(2,4)} ---------------- (A)

  $R1^C$ = {(4,1),(3,2),(2,3),(1,4)}

  $R2^C$= {(2,1),(3,2),(4,3)}

  $R2^C. R1^C$ = {(2,4),(3,3),(4,2)} ---------------(B)

  Therefore from (A) & (B) we get, **$(R1.R2)^C = R2^C. R1^C$.**

- **Example 8:** Let R = {(1, 1), (2, 1), (3, 2), (4, 3)}. Find the powers R^n, n = 2, 3,4,……

  Solution: We have

R = {(1, 1), (2, 1), (3, 2), (4, 3)}.

R^2 = R ∘ R = [{(1, 1), (2, 1), (3, 2), (4, 3)}] [{(1, 1), (2, 1), (3, 2), (4, 3)}]

R^2 = {(1, 1), (2, 1), (3, 1), (4, 2)}.

R^3 = R^2 ∘ R = [{(1, 1), (2, 1), (3, 1), (4, 2)}] [{(1, 1), (2, 1), (3, 2), (4, 3)}]

R^3 = {(1, 1), (2, 1), (3, 1), (4, 1)}.

R^4 = R^3 ∘ R = [{(1, 1), (2, 1), (3, 1), (4, 1)}] [{(1, 1), (2, 1), (3, 2), (4, 3)}]

R^4 = {(1, 1), (2, 1), (3, 1), (4, 1)}

- **Example 9:** R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.

  Solution: R^1 = {(1,2),(2,3),(2,4), (3,3)}

  R^2 = {(1,3), (1,4), (2,3), (3,3)}

  R^3 = {(1,3), (2,3), (3,3)}

  R^4 = {(1,3), (2,3), (3,3)}

- **Example 10:** Let R be the relation on the set {1, 2, 3, 4, 5} containing the ordered pairs (1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 1), (3, 4), (3, 5), (4, 2), (4, 5), (5, 1), (5, 2), and (5, 4). Find a) R^2. b) R^3. c) R^4. d) R^5.

  Solution: R^1 =

  R^2 =

  R^3 =

  R^4 =

  R^5 =

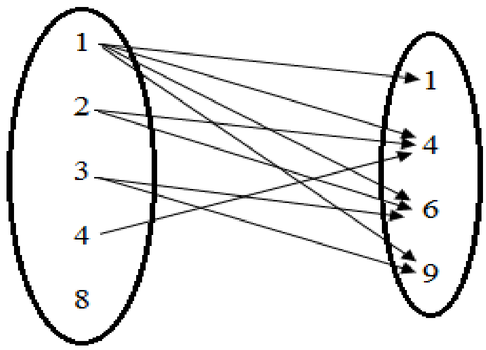
❖ Representation of Relations:

i. Ordered Pair

- The first elements in the ordered pairs (the x-values), form the domain. The second elements in the ordered pairs (the y-values), form the range. Only the elements "used" by the relation constitute the range.
- **Example 11:** Let set A = (1,2,3,4,8) and set B = (1,4,6,9). R={(x,y)| y is divisible by x}.

  Solution: The relation R consists of the ordered pairs:

  R={(1,1),(1,4),(1,6),(1,9),(2,4),(2,6),(3,6),(3,9),(4,4)}



ii. Tabular Representation

- Tabular Representation for R={(1,1),(1,4),(1,6),(1,9),(2,4),(2,6),(3,6),(3,9),(4,4)}

	1	4	6	9
1	*	*	*	*
2		*	*	
3			*	*
4		*		
8				

iii. Matric Representation

- A relation between finite sets can be represented using a zero–one matrix.
- Suppose that R is a relation from A = {a1, a2 , . . . , am } to B = {b1 , b2 , . . . , bn }. (Here the elements of the sets A and B have been listed in a particular, but arbitrary, order. Furthermore, when A = B we use the same ordering for A and B.)
- The relation R can be represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

- i.e. the zero–one matrix representing R has 1 as its (i,j) entry when a_i is related to b_j, and 0 in this position if a_i is not related to b_j . (Such a representation depends on the orderings used for A and B.)
- Matrix Representation for R={(1,1),(1,4),(1,6),(1,9),(2,4),(2,6),(3,6),(3,9),(4,4)} is:

$$M_R = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

- **Representing Relations using Zero–One Matrices:**
  - ✓ Let A = [aij] and B = [bij] be m × n zero–one matrices.
  - ✓ The Join of A and B is the zero–one matrix with (i, j)th entry aij ∨ bij. The join of A and B is denoted by A ∨ B.

- ✓ The Meet of A and B is the zero–one matrix with $(i, j)^{th}$ entry aij ∧ bij. The meet of A and B is denoted by A ∧ B.
- ✓ Example: Let A = {1,2,3} and B = {u,v} and

  R1 = {(1,u), (2,u), (2,v), (3,u)} R2 = {(1,v),(3,u),(3,v)}

$$M_{R1} = \begin{matrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{matrix} \qquad M_{R2} = \begin{matrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{matrix} \qquad M_{(R1 \vee R2)} = \begin{matrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{matrix}$$

$$M_{R1} = \begin{matrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{matrix} \qquad M_{R2} = \begin{matrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{matrix} \qquad M_{(R1 \wedge R2)} == \begin{matrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{matrix}$$

- **Matrix Relation for Union and Intersection operation:**
  - ✓ The Boolean operations join and meet can be used to find the matrices representing the **union and the intersection of two relations**. Suppose that R_1 and R_2 are relations on a set A represented by the matrices M_{R1} and M_{R2}, respectively.
  - ✓ The matrix representing the union of these relations has a 1 in the positions where either M_{R1} or M_{R2} has a 1.
  - ✓ The matrix representing the intersection of these relations has a 1 in the positions where both M_{R1} andM_{R2} have a 1.
  - ✓ Thus, the matrices representing the union and intersection of these relations are

$$M_{R1 \cup R2} = M_{R1} \vee M_{R2} \qquad \text{and} \qquad M_{R1 \cap R2} = M_{R1} \wedge M_{R2}$$

- **Example 12:** Suppose that the relations R1 and R2 on a set A are represented by the matrices. What are the matrices representing R1 ∪ R2 and R1 ∩ R2?

$$M_{R1} = \begin{matrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \qquad M_{R2} = \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix}$$

Solution: The matrices of these relations are:

$$M_{R1 \cup R2} = M_{R1} \vee M_{R2} = \begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{matrix}$$

$$M_{R1 \cap R2} = M_{R1} \wedge M_{R2} == \begin{matrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

- **Matrix for Composite of Relations:**
  - ✓ The matrix can be found using the Boolean product of the matrices for these relations.
  - ✓ The Boolean product denoted by ⊙,of an m-by-n matrix (a_{ij}) and n-by-p matrix (b_{jk}) of 0s and 1s is an m-by-p matrix (m_{ik}) where

$m_{ik} = 1$, if $a_{ij} = 1$ and $b_{jk} = 1$ for some k=1,2,...,n
0, otherwise.

- ✓ Suppose that R is a relation from A to B and S is a relation from B to C. From the definition of the Boolean product, this means that

$$M_{S \circ R} = M_R \odot M_S$$

- ✓ The matrix representing the composite of two relations can be used to find the matrix for M_{R^n}. In particular

$$M_R n = M_R^{[n]}$$

- **Example 13:** Let A = {1,2}, {1,2,3} and C = {a,b}.
  R = {(1,2),(1,3),(2,1)} is a relation from A to B.
  S = {(1,a),(3,b),(3,a)} is a relation from B to C.
  R.S = {(1,b),(1,a),(2,a)}

$M_R = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix}$ $M_S = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$ $M_R \odot M_S = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix} \odot \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$

(0 ∧ 1) ∨ (1 ∧ 0) ∨ (1 ∧	(0 ∧ 0) ∨ (1 ∧ 0) ∨ (1 ∧ 1)
(1 ∧ 1) ∨ (0 ∧ 0) ∨ (0 ∧	(1 ∧ 0) ∨ (0 ∧ 0) ∨ (0 ∧ 1)

0 ∨ 0 ∨ 1	0 ∨ 0 ∨ 1
1 ∨ 0 ∨ 0	0 ∨ 0 ∨ 0

$M_{S \circ R} = M_R \odot M_S = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$

- **Example 14:** Find the matrix representing the relations S∘R, where the matrices representing R and S are

$M_R = \begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$ $M_S = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{matrix}$

$M_R \odot M_S = \begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \odot \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{matrix}$

(1 ∧ 0) ∨ (0 ∧ 0) ∨ (1 ∧ 1)	(1 ∧ 1) ∨ (0 ∧ 0) ∨ (1 ∧ 0)	(1 ∧ 0) ∨ (0 ∧ 1) ∨ (1 ∧ 1)
(1 ∧ 0) ∨ (1 ∧ 0) ∨ (0 ∧ 1)	(1∧ 1) ∨ (1 ∧ 0) ∨ (0 ∧ 0)	(1 ∧ 0) ∨ (1 ∧ 1) ∨ (0 ∧ 1)
(0 ∧ 0) ∨ (0 ∧ 0) ∨ (0 ∧ 1)	(0 ∧ 1) ∨ (0 ∧ 0) ∨ (0 ∧ 0)	(0 ∧ 0) ∨ (0 ∧ 1) ∨ (0 ∧ 1)

0 ∨ 0 ∨ 1	1 ∨ 0 ∨ 0	0 ∨ 0 ∨ 1

$0 \vee 0 \vee 0$	$1 \vee 0 \vee 0$	$0 \vee 1 \vee 0$
$0 \vee 0 \vee 0$	$0 \vee 0 \vee 0$	$0 \vee 0 \vee 0$

$$M_{S \circ R} = M_R \odot M_S = \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{matrix}$$

- **Example 15:** Find the matrix representing the relation R^2, where the matrix representing R is

$$M_R = \begin{matrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad \text{Solution: } M_R2 = M_R^{[2]} = \begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{matrix}$$

- **Example 16:** Let R be the relation represented by the matrix. Find the matrices that represent a) R^2. b) R^3. c) R^4.

$$M_R = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$$

Solution: We compute the Boolean powers of M_R; thus

$$M_R2 = M_R^{[2]} = M_R \odot M_R$$

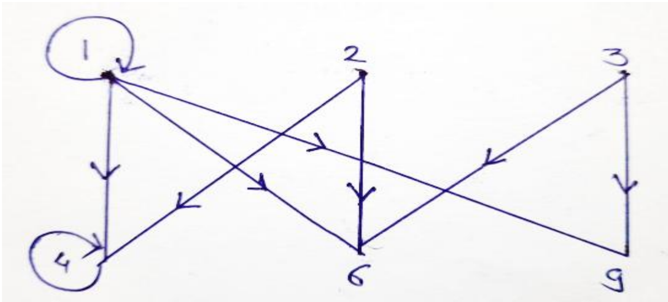
$$M_R3 = M_R^{[3]} = M_R^{[2]} \odot M_R$$

$$M_R4 = M_R^{[4]} = M_R^{[3]} \odot M_R$$

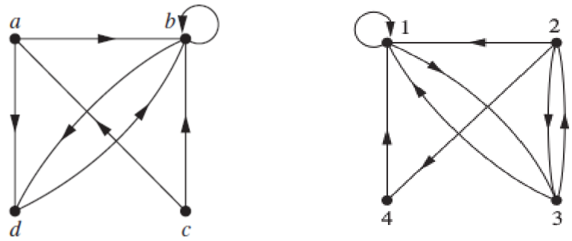
$$\text{a) } R^2 = \begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix} \qquad \text{b) } R^3 = \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} \qquad \text{c) } R^4 = \begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

iv. Digraph Representation

- A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.
- An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called a loop.
- Digraph representation for R={(1,1),(1,4),(1,6),(1,9),(2,4),(2,6),(3,6),(3,9),(4,4)} is:



- The directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b) is displayed in figure (a) below.
- The directed graph of the relation R = {(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)} on the set {1, 2, 3, 4} is shown in figure (b) below:



(a) **(b)**

- **Example 17:** Let A = {1,2,3,4}. Define a $R_{\neq}$ b if and only if a ≠ b. What is Inverse Complement of $R_{\neq}$ and Also representation of given relation $R_{\neq}$.

  Solution: $R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

R	1	2	3	4
1		x	x	x
2	x		x	x
3	x	x		x
4	x	x	x	

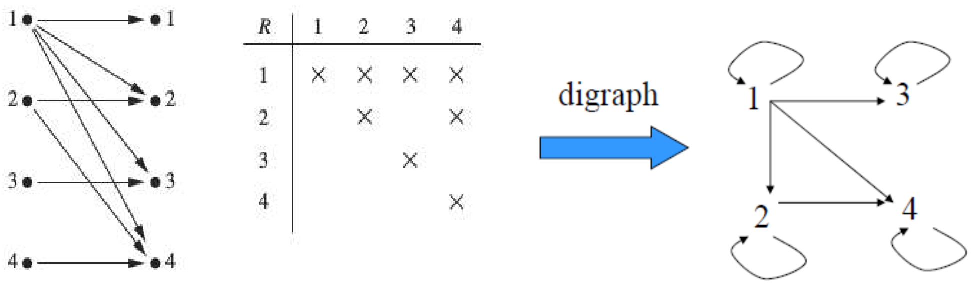
- **Example 18:** List the ordered pairs in the relation R from A = {0, 1, 2, 3, 4} to B = {0, 1, 2, 3}, where (a, b) ∈ R if and only if i) a = b. ii) a + b = 4. iii) a > b. iv) a | b.

  Solution:

  i. {(0,0), (1,1), (2,2), (3,3)}
  ii. {(1,3),(2,2),(3, 1), (4,0)}
  iii. {(1,0), (2,0), (2,1),(3,0), (3,1),(3,2),(4,0), (4,1), (4,2), (4,3)}
  iv. a | b means that b is a multiple of a (a is not allowed to be 0).
  {(1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 0), (3, 3), (4, 0)}.

- **Example 19:** Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a divides b}?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b. R = {(1,1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}.



- **Example 20:** Consider these relations on the set of integers:

  R1 = {(a, b) | a ≤ b},

  R2 = {(a, b) | a > b},

  R3 = {(a, b) | a = b or a = −b},

  R4 = {(a, b) | a = b},

  R5 = {(a, b) | a = b + 1},

  R6 = {(a, b) | a + b ≤ 3}.

  Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1, −1), and (2, 2)?

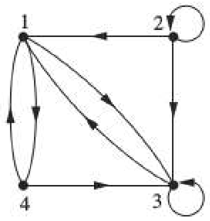
  Solution: The pair (1, 1) is in R1, R3, R4 , and R6; (1,2) is in R1 and R6; (2,1) is in R2, R5, and R6; (1, −1) is in R2 , R3 , and R6 ; and finally, (2,2) is in R1 , R3 , and R4 .

- **Example 21:** Let A = {a1, a2, 3} and B = {b1, b2, b3, b4, b5}. Which ordered pairs are in the relation R represented by the matrix.

$$M_R = \begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{matrix}$$

  Solution: R = {(a1,b2), (a2,b1), (a2,b3), (a2,b4), (a3,b1), (a3,b3), (a3,b5)}.

- **Example 22:**What are the ordered pairs in the R represented by the directed graph:



  Solution: The ordered pairs (x, y) in the relation are

  R = {(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)}.

  Each of these pairs corresponds to an edge of the directed graph, with (2, 2) and (3, 3) corresponding to loops.

- **Example 23:** How many different relations are there from a set with m elements to a set with n elements?

  Solution: There are mn elements of the set A x B, if A is a set with m elements and B is a set with n elements. A relation from A to B is a subset of A x B. Thus the question asks for the number of subsets of the set A x B, which has mn elements.

  Therefore by the product rule, it is $\mathbf{2^{mn}}$**.**

- **Example 24:** How many relations are there on a set with n elements.

- **Theorem:** The number of binary relations on a set A, where | A | = n is:

$$2^{n^2}$$

- **Proof:**
- If | A | = n then the cardinality of the Cartesian product | A x A | = n^2.
- R is a binary relation on A if R $\subseteq$ A x A (that is, R is a subset of A x A).
- The number of subsets of a set with k elements : 2^k
- The number of subsets of A x A is : $2^{|AxA|} = 2^{n^2}$

- **Example**: Let A = {1,2}
- What is A x A = **{(1,1),(1,2),(2,1),(2,2)}**
- **List of possible relations (subsets of A x A):**
- $\varnothing$ 1
- {(1,1)} {(1,2)} {(2,1)} {(2,2)} 4
- {(1,1), (1,2)} {(1,1),(2,1)} {(1,1),(2,2)} 6
  {(1,2),(2,1)} {(1,2),(2,2)} {(2,1),(2,2)}
- {(1,1),(1,2),(2,1)} {(1,1),(1,2),(2,2)} 4
  {(1,1),(2,1),(2,2)} {(1,2),(2,1),(2,2)}
- {(1,1),(1,2),(2,1),(2,2)} 1

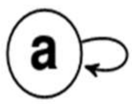
} **16**

- Use formula: $\mathbf{2^4 = 16}$

❖ Properties of a Relation

1. **Reflexive Relation:** Relation R on a set A is called **reflexive** if (a,a)$\in$ R for every element a $\in$ A.

   ✓ A relation is **reflexive** if, we observe that for all values a: aRa

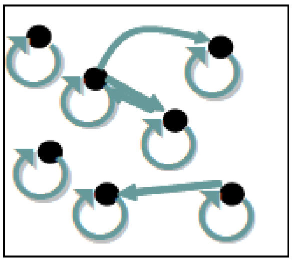
   

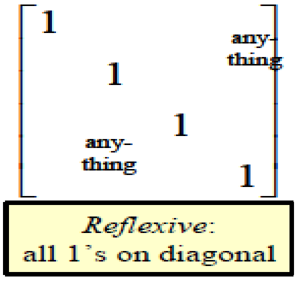
   ✓ In other words, all values are related to themselves.

   ✓ A relation R is said to be **not reflexive** if there exist at least one element a $\in$ A such that (a,a) $\notin$ R. i.e. a $\not R$ a

   ✓ The relation of equality, "=" is reflexive. In a reflexive relation, we have arrows for all values in the domain pointing back to themselves.

- ✓ Note that ≤ and ≥ is reflexive (a ≤ a for any a in R). On the other hand, the relation < or > is not reflexive (a < a is false for any a in R).
- ✓ A Relation R is reflexive if all the elements on the main diagonal of M_R are equal to 1, and the elements off the main diagonal can be either 0 or 1.
- ✓ Every Node has a self-loop.





- ✓ Example: Assume relation R_{div} ={ (a b), if a | b} on A = {1,2,3,4}. Is R_{div} reflexive?

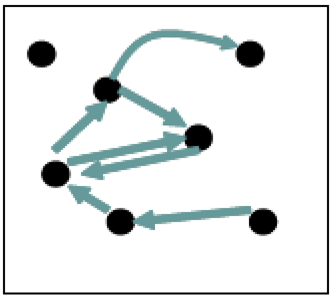
$$R_{div} = \{(1,1), (1,2), (1,3), (1,4),(2,2), (2,4), (3,3), (4,4)\}$$

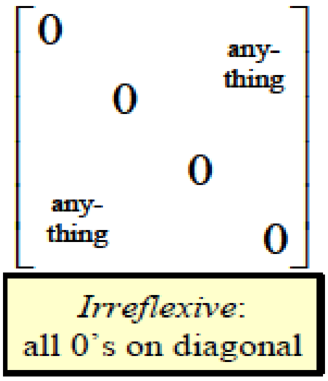
Answer: R_{div} reflexive because (1,1), (2,2), (3,3), and (4,4) ∈ A.

$$MR_{div} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

- ✓ Example: Relation R_{fun} on A = {1,2,3,4} defined as: R_{fun} = {(1,2),(2,2),(3,3)}. Answer: No. R_{fun} is not reflexive relation since (1,1) and (4,4) ∉ R_{fun}.

2. **Irreflexive Relation:** Relation on a set A is called **irreflexive** if (a, a) ∉R for every element a ∈ A.
   - ✓ A relation is **Irreflexive** if, we observe that for all values a: aRa does not hold
   - ✓ The relation R = {(a, b), (b, a)} on set X = {a, b} is irreflexive.
   - ✓ A relation R is reflexive if and only if Matrix representation ($M_{R)}$ has 0 in every position on its main diagonal.
   - ✓ No node has a self-loop.





- ✓ Example: Assume relation $R_{\neq}$ on A = {1,2,3,4}, such that $aR_{\neq}b$ if and only if a ≠ b.

$R_{\neq}$ = {(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

Answer: $R_{\neq}$ irreflexive Because (1,1),(2,2),(3,3) and (4,4) $\notin R_{\neq}$.

- ✓ Example: Let A = {1,2,3,4} and R_{fun}= {(1,2),(2,2),(3,3)}.Is R_{fun} irreflexive?

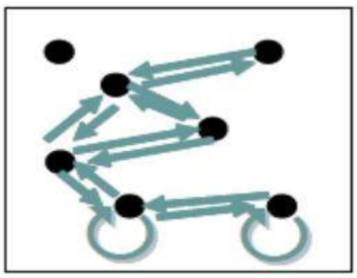
  Answer: No, Because (2,2) and (3,3) $\in R_{fun}$.

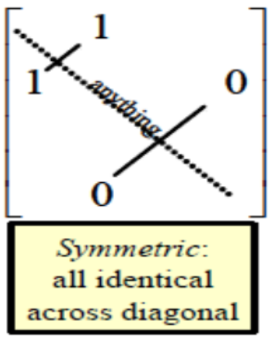
3. **Symmetric Relation:** Relation R on a set A is called symmetric if (b, a) $\in$ R whenever (a, b) $\in$ R $\forall$ a, b$\in$ A. i. e. $\forall$ (a, b) ∈ A (a,b) ∈ R → (b,a) ∈ R.

- ✓ A relation is **symmetric** if, we observe that for all values of a & b: a R b implies b R a
- ✓ The relation of equality is symmetric. If x = y, we can also write that y = x also.
- ✓ Relation R is symmetric if and only if **mij = mji**, for all pairs of integers i and j.
- ✓ Recalling the definition of the transpose of a matrix, we see that R is symmetric if and only if $\mathbf{M_R=(M_R)^t}$, that is, if M_R is a symmetric matrix.
- ✓ Neither ≤ nor < is symmetric (2 ≤ 3 and 2 < 3 but neither 3 ≤ 2 nor 3 < 2 is true).
- ✓ Example: Let R_{div} ={(a b), if a | b} on A = {1,2,3,4}. Is R_{div} symmetric?

  R_{div} = {(1,1), (1,2), (1,3), (1,4),(2,2), (2,4), (3,3), (4,4)}

  Answer: No. It is not symmetric since (1,2) $\in$ R but (2,1) $\notin$ R.





- ✓ Example: Assume relation $R_{\neq}$ on A={1,2,3,4}, such that a $R_{\neq}$ b if and only if a ≠ b.

  $R_{\neq}$ = {(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

  Answer: $R_{\neq}$ symmetric, If (a,b) ∈ $R_{\neq}$ →(b,a) ∈ $R_{\neq}$

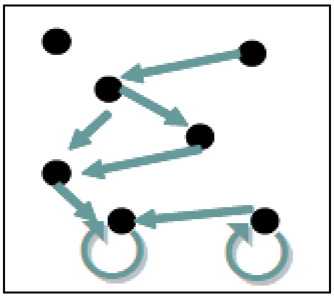
$$M_R = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

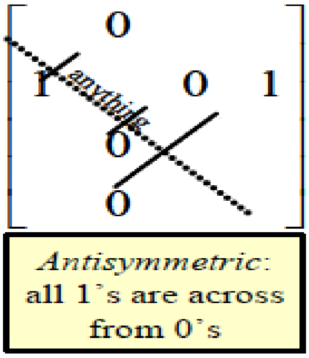
- ✓ Example: Relation R_{fun} on A = {1,2,3,4} defined as: R_{fun} = {(1,2),(2,2),(3,3)}. Answer: No. It is not symmetric relation since (1,2) $\in R_{fun}$ and (2,1) $\notin R_{fun}$.

4. **Anti-Symmetric Relation:** A relation R on a set A such that for all a, b ∈ A, if (a, b) ∈ R and (b, a) ∈ R, then a = b is called **antisymmetric.**

- ✓ The matrix of an antisymmetric relation has the property that if mij =1 then mji = 0 for i ≠ j. Or, in other words, either mij = 0 or mji = 0 when i ≠ j.

- ✓ No link is Bidirectional.





- ✓ Example: Let A = {1,2,3,4} and R = {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)} Is R antisymmetric?

  Answer: It is not antisymmetric since it includes both (2, 3) and (3, 2), but 2 ≠ 3.
- ✓ Example: Let A = {1,2,3,4} and R_{fun}= {(1,2),(2,2),(3,3)}.Is R_{fun} antisymmetric?

  Answer: Yes R_{fun} antisymmetric since there are no cases of (a, b) and (b, a) in R_{fun}.

$$MR_{fun} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

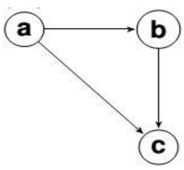
Imp Notes:

- ✓ A relation is **symmetric** if and only if a is related to b implies that b is related to a.
- ✓ A relation is **antisymmetric** if and only if there are no pairs of distinct elements a and b with a related to b and b related to a. That is, the only way to have a related to b and b related to a is for a and b to be the same element.
- ✓ The terms symmetric and antisymmetric are **not opposites**, because a relation can have both of these properties or may lack both of them. A relation cannot be both symmetric and antisymmetric if it contains some pair of the form (a, b), where a = b.

5. **Asymmetric Relation:** Relation R on a set A is called **asymmetric** if (a, b)∈R implies that (b, a)∉ R ∀ a,b∈A.

6. **Transitive Relation:** Relation R on a set A is called **transitive** if whenever (a, b)∈ R and (b, c)∈R, then (a, c)∈R for a, b, c∈A.

- ✓ The relation R = {(1, 2), (2, 3), (1, 3)} on set A = {1, 2, 3} is transitive.



- ✓ The relation greater-than ">" is transitive. If x > y, and y > z, then it is true that x > z. This becomes clearer when we write down what is happening into words. x is greater than y and y is greater than z. So x is greater than both y and z.
- ✓ The relation is-not-equal "≠" is not transitive. If x ≠ y and y ≠ z then we might have x = z or x ≠ z (for example 1 ≠ 2 and 2 ≠ 3 and 1 ≠ 3 but 0 ≠ 1 and 1 ≠ 0 and 0 = 0).
- ✓ Example: Assume relation R_{div} = {(a b), if a|b} on A = {1,2,3,4}. Is R_{div} transitive?

  R_{div} = {(1,1), (1,2), (1,3), (1,4),(2,2), (2,4), (3,3), (4,4)}

  Answer: Yes R_{div} is transitive relation.
- ✓ Example: Assume relation $R_{\neq}$ on A = {1,2,3,4} such that $aR_{\neq}b$ if and only if a ≠ b.

  $R_{\neq}$ = {(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

  Answer: $R_{\neq}$ is not transitive relation since (1,2) ∈ $R_{\neq}$ & (2,1) ∈ $R_{\neq}$ but (1,1) ∉ $R_{\neq}$.

7. **Equivalence Relation**: A relation is an **equivalence relation** if it is reflexive, symmetric, and transitive.
   - ✓ Two elements a & b that are related by an equivalence relation are called **equivalent.**
   - ✓ The notation **a ∼ b** or **"a ≡ b"** is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.
   - ✓ Example: The relation R = {(1, 1), (2, 2), (3, 3), (1, 2), (2,1), (2,3), (3,2), (1,3), (3,1)} on set A = {1, 2, 3} is an equivalence relation since it is reflexive, symmetric, and transitive.

❖ **Equivalence Classes**

- ✓ Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a.
- ✓ The equivalence class of a with respect to R is denoted by $[a]_R$.
- ✓ When only one relation is under consideration, we can delete the subscript R and write [a] for this equivalence class. In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is :

$$[a]_R = \{s \in A \mid (a, s) \in R\}.$$

- ➢ **Example 25:** For each of these relations on the set {1,2,3,4} decide whether it is reflexive, symmetric, antisymmetric, and transitive.
  - i. {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}
  - ii. {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}
  - iii. {(2, 4), (4, 2)}
  - iv. {(1, 2), (2, 3), (3, 4)}
  - v. {(1, 1), (2, 2), (3, 3), (4, 4)}

vi. {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

Solution:

i. {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}

- ✓ The relation is not reflexive, since it does not include, for instance (1, 1) and (4, 4).
- ✓ It is not symmetric, since it includes, for instance, (2, 4) but not (4, 2). And we have (3, 4), but not have (4, 3).
- ✓ It is not antisymmetric since it includes both (2, 3) and (3, 2), but 2 ≠ 3.
- ✓ It is transitive. To see this we have to check that whenever it includes (a,b) and (b,c), then it also includes (a, c). We can ignore the element 1 since it never appears. If (a, b) is in this relation, then by inspection we see that a must be either 2 or 3. But (2, c) and (3, c) are in the relation for all c ≠ 1; thus (a, c) has to be in this relation whenever (a, b) and (b, c) are. This proves that the relation is transitive.

ii. {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

- ✓ The relation is reflexive, since all the pairs (1, 1), (2, 2), (3, 3), and (4, 4) are in it.
- ✓ It is clearly symmetric, the only nontrivial case to note being that both (1, 2) and (2, 1) are in the relation.
- ✓ It is not antisymmetric because both (1, 2) and (2, 1) are in the relation.
- ✓ It is transitive; the only nontrivial cases to note are that since both (1, 2) and (2, 1) are in the relation, we need to have (and do have) both (1, 1) and (2, 2) included as well.

iii. {(2, 4), (4, 2)}

- ✓ The relation clearly is not reflexive and clearly is symmetric. It is not antisymmetric since both (2, 4) and (4, 2) are in the relation.
- ✓ It is not transitive, since although (2,4) and (4,2) are in the relation, (2,2) is not.

iv. {(1, 2), (2, 3), (3, 4)}

- ✓ This relation is clearly not reflexive. It is not symmetric, since, for instance, (1, 2) is included but (2, 1) is not.
- ✓ It is antisymmetric, since there are no cases of (a, b) and (b, a) both being in the relation.
- ✓ It is not transitive, since although (1, 2) and (2, 3) are in the relation, (1, 3) is not

v. {(1, 1), (2, 2), (3, 3), (4, 4)}

- ✓ This relation is clearly reflexive and symmetric.
- ✓ It is trivially antisymmetric since there are no pairs (a, b) in the relation with a ≠ b.

vi. {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

- ✓ This relation is clearly not reflexive.
- ✓ The presence of (1,4) and absence of (4, 1) shows that it is not symmetric.

- ✓ The presence of both (1, 3) and (3, 1) shows that it is not antisymmetric.
- ✓ It is not transitive; both (2, 3) and (3, 1) are in the relation, but (2, 1) is not.

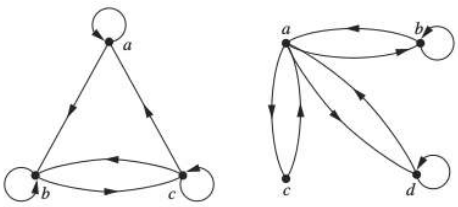
➢ **Example 26:** Suppose that the relation R on a set is represented by the matrix.

$$M_R = \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{matrix}$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements of this matrix are equal to 1, R is reflexive. Moreover, because M_R is symmetric, it follows that R is symmetric. It is also easy to see that R is not antisymmetric.

➢ **Example 27:** Determine whether the relations for the directed graphs shown in figure below are reflexive, symmetric, antisymmetric, and/or transitive.



(a) Directed graph of *R* (b) Directed graph of *S*

From Fig (a) for Directed Graph of R

- ✓ Relation R is reflexive because there are loops at every vertex of the graph of R.
- ✓ Relation R is neither symmetric nor antisymmetric because there is an edge from a to b but not one from b to a, but there are edges in both directions connecting b and c.
- ✓ Relation R is not transitive because there is an edge from a to b and an edge from b to c, but no edge from a to c.

From Fig (b) for Directed Graph of S.

- ✓ Relation S is not reflexive because loops are not present at all the vertices of the directed graph of S.
- ✓ It is symmetric and not antisymmetric, because every edge between distinct vertices is accompanied by an edge in the opposite direction.
- ✓ It is also not hard to see from the directed graph that S is not transitive, because (c, a) and (a, b) belong to S, but (c, b) does not belong to S.

➢ **Example 28:**

i. Let R be the relation on the set of real numbers such that aRb if and only if a−b is an integer. Is R an equivalence relation?

Solution: Because a−a =0 is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive.

Now suppose that aRb. Then a−b is an integer, so b−a is also an integer. Hence, bRa. It follows that R is symmetric.

If aRb and bRc, then a−b and b–c are integers. Therefore, a−c=(−b)+(b−c) is also an integer. Hence, aRc. Thus, R is transitive.

Consequently, R is an equivalence relation.

ii. Consider the relation on A= {1,2,3,4,5,6}. R= {(i, j) | i - j = 2}. Is R an Equivalences Relation?

Solution: R={(1,3),(3,1),(2,4),(4,2),(3,5),(5,3),(4,6),(6,4)}

Relation R is not Reflexive as (2,2) is not belong to R. Relation R is Symmetric and not Transitive Relation. Therefore is not a Equivalences Relation.

iii. Consider the relation on A= {1,2,3,4,5,6,7}. R= {(x, y) | x - y is divisible by 3}. Show that R an Equivalences Relation and draw Digraph of R.

Solution: We know x-x = 0 is divisible by 3

x R x, For Every x belong to A, R is Reflexive Relation.

As, xRy = x-y is divisible by 3

= y-x is also divisible by 3

= y-z is divisible by 3

= y R x for (x,y) ∈ A

Therefore R is Symmetric Relation.

As, xRy and yRz = x - y and y - z are divisible by 3

= (x – y) + (y – z) are divisible by 3

= (x - z) is divisible by 3

= xRz

Therefore R is Transitive Relation.

Therefore R is Equivalences Relation.

- **Example 29:** Example: Which of these relations on {0, 1, 2, 3} are equivalence relations? Determine the properties of an equivalence relation that the others lack.

  a) {(0, 0), (1, 1), (2, 2), (3, 3)}

  b) {(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}

  c) {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}

  d) {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

  e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

Solution: a) This is an equivalence relation; it is easily seen to have all three properties. The equivalence classes all have just one element.

b) This relation is not reflexive since the pair (1, 1) is missing. It is also not transitive, since the pairs (0, 2) and (2, 3) are there, but not (0, 3).

c) This is an equivalence relation. The elements 1 and 2 are in the same equivalence class; 0 and 3 are each in their own equivalence class.

d) This relation is reflexive and symmetric, but it is not transitive. The pairs (1, 3) and (3, 2) are present, but not (1, 2).

e) This relation would be an equivalence relation were the pair (2, 1) present. As it is, its absence makes the relation neither symmetric nor transitive.

- **Example 30:** Consider the following relations on {1, 2, 3, 4}:

  i. R1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)},

  ii. R2 = {(1, 1), (1, 2), (2, 1)},

  iii. R3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)},

  iv. R4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)},

  v. R5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)},

  vi. R6 = {(3, 4)}.

  Which of these relations are reflexive, symmetric, antisymmetric and transitive?

  Solution:

Relation	Reflexive	Symmetric	Antisymmetric	Asymmetric	Transitive
R1				Y	
R2		Y			
R3	Y	Y			
R4			Y	Y	Y
R5	Y		Y		Y
R6			Y	Y	Y

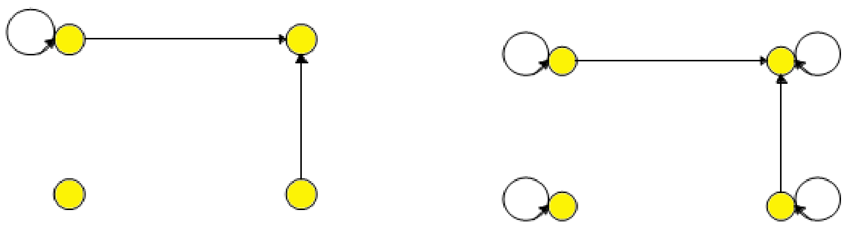
Closures of Relations:

- **Definition:** The closure of a relation R with respect to property P is the relation obtained by adding the minimum number of ordered pairs to R to obtain property P.
- In terms of the digraph representation of R
  - ✓ To find the reflexive closure - add loops.
  - ✓ To find the symmetric closure - add arcs in the opposite direction.
  - ✓ To find the transitive closure - if there is a path from a to b, add an arc from a to b.

- **Note:** Reflexive and symmetric closures are easy. Transitive closures can be very complicated.

1. Reflexive Closures

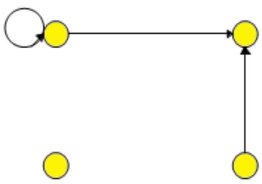
- ✓ The relation R = {(1, 1), (1, 2), (2, 1), (3, 2)} on the set A = {1, 2, 3} is not reflexive.
- ✓ How can we produce a reflexive relation containing R that is as small as possible? This can be done by adding (2, 2) and (3, 3) to R, because these are the only pairs of the form (a, a) that are not in R.
- ✓ This new relation contains R. Furthermore, any reflexive relation that contains R must also contain (2, 2) and (3, 3).
- ✓ Because this relation contains R, is reflexive, and is contained within every reflexive relation that contains R, it is called the **reflexive closure of R.**
- ✓ As this example illustrates, given a relation R on a set A, the reflexive closure of R can be formed by adding to R all pairs of the form (a, a) with a ∈ A, not already in R.
- ✓ The addition of these pairs produces a new relation that is reflexive, contains R, and is contained within any reflexive relation containing R.
- ✓ **We see that the reflexive closure of R equals R ∪ Δ, where Δ= {(a, a) | a ∈ A} is the diagonal relation on A.**
- ✓ **Add loops to all vertices on the digraph representation of R.**
- ✓ **Put 1's on the diagonal of the connection matrix of R.**

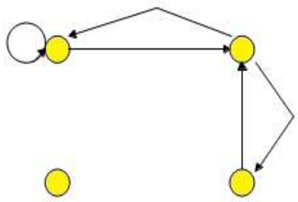


2. Symmetric Closure

- ✓ The relation {(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)} on {1, 2, 3} is not symmetric.
- ✓ How can we produce a symmetric relation that is as small as possible and contains R? To do this, we need only add (2, 1) and (1, 3), because these are the only pairs of the form (b, a) with (a, b) ∈ R that are not in R.
- ✓ This new relation is symmetric and contains R. Furthermore, any symmetric relation that contains R must contain this new relation, because a symmetric relation that contains R must contain (2, 1) and (1, 3). Consequently, this new relation is called the **symmetric closure of R.**

- ✓ As this example illustrates, the symmetric closure of a relation R can be constructed by adding all ordered pairs of the form (b, a), where (a, b) is in the relation, that are not already present in R.
- ✓ Adding these pairs produces a relation that is symmetric, that contains R, and that is contained in any symmetric relation that contains R.
- ✓ **The symmetric closure of a relation can be constructed by taking the union of a relation with its inverse that is, $R \cup R^{-1}$ is the symmetric closure of R, where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.**
- ✓ **Reverse all the arcs in the digraph representation of R.**
- ✓ **Take the transpose M^T of the connection matrix M of R.**





- **Example 31:** Let R be the relation on the set A = {0, 1, 2, 3} R= {(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0)}. Find reflexive closure and symmetric closure of R.

  Solution: The reflexive closure $R = R \cup \Delta$, where $\Delta = \{(a, a) \mid a \in A\}$.

  R = {(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0)}

  Therefore Δ = {(0, 0), (1, 1), (2, 2), (3, 3)}.

  $R = R \cup \Delta$ = {(0, 0),(0, 1),(1, 1),(1, 2),(2, 0),(2, 2),(3, 0),(3, 3)}.

  The symmetric closure of $R = R \cup R^{-1}$, where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

  R = {(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0)}

  R^{-1} = {(1, 0), (1, 1), (2, 1), (0, 2), (2, 2), (0, 3)}

  $R = R \cup R^{-1}$ = { (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (2, 0),(2 ,1), (2, 2), (3, 0)}.

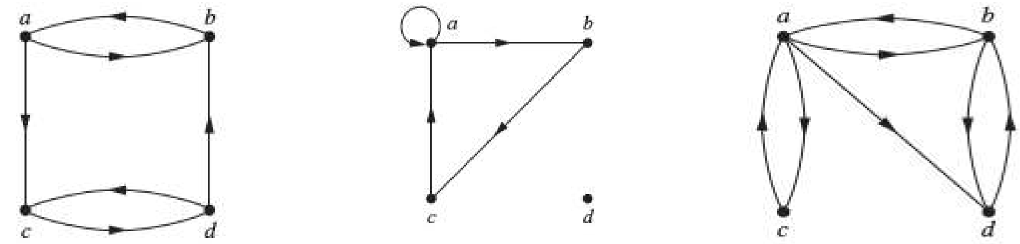
- **Example 32:** Let R be the relation {(a, b) | a divides b} on the set of integers. What is the symmetric closure of R?

  Solution: To form the symmetric closure we need to add all the pairs (b, a) such that (a, b) is in R.

  In this case, that means that we need to include pairs (b, a) such that a divides b, which is equivalent to saying that we need to include all the pairs (a, b) such that b divides a.

  Thus the closure is { (a, b) | a divides b or b divides a }

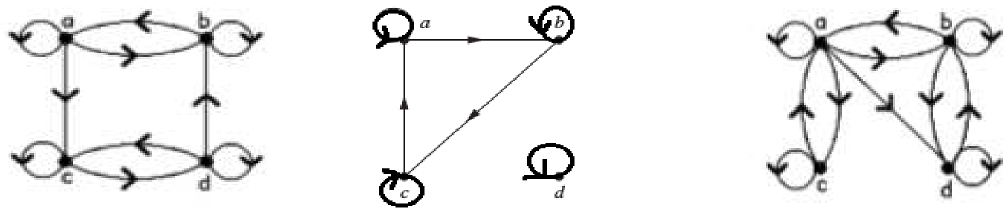
- **Example 33:** Draw the directed graph of the reflexive closure and Symmetric closure of the relations with the directed graph shown.



Solution: Reflexive closure:

Add loops to all vertices on the digraph representation of R.

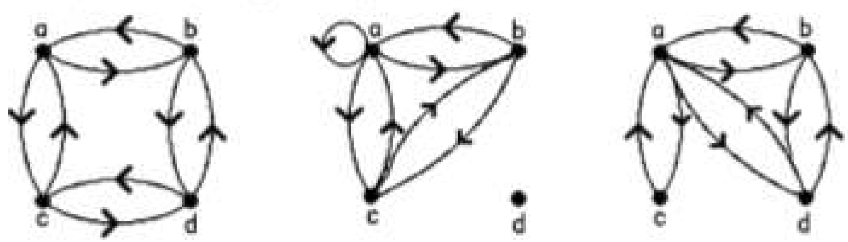
Put 1's on the diagonal of the connection matrix of R.



Symmetric closure:

Reverse all the arcs in the digraph representation of R

Take the transpose M^T of the connection matrix M of R



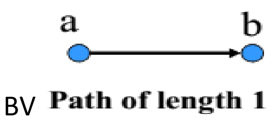
3. Transitive Closure

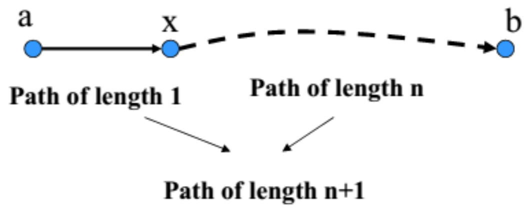
- ✓ Suppose that a relation R is not transitive.
- ✓ How can we produce a transitive relation that contains R such that this new relation is contained within any transitive relation that contains R?
- ✓ Can the transitive closure of a relation R be produced by adding all the pairs of the form (a, c), where (a, b) and (b, c) are already in the relation?
- ✓ Consider the relation R = {(1, 3), (1, 4), (2, 1), (3, 2)} on the set {1, 2, 3, 4}.
- ✓ This relation is not transitive because it does not contain all pairs of the form (a, c) where (a, b) and (b, c) are in R.
- ✓ The pairs of this form not in R are (1, 2), (2, 3), (2, 4), and (3, 1).
- ✓ Adding these pairs does not produce a transitive relation, because the resulting relation contains (3, 1) and (1, 4) but does not contain (3, 4).
- ✓ This shows that constructing the transitive closure of a relation is more complicated than constructing either the reflexive or symmetric closure.

- ✓ The transitive closure of a relation can be found by adding new ordered pairs that must be present and then repeating this process until no new ordered pairs are needed.

➢ Paths in Directed Graphs

- ✓ **Theorem:** Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if (a, b)∈ R^n.
- ✓ **Proof:** We will use mathematical induction. By definition, there is a path from a to b of length one if and only if (a, b) ∈ R, so the theorem is true when n=1.





There is a path of length n+1 from a to b if and only if there exists an x ∈ A, such that (a, x) ∈ R (a path of length 1) and (x, b) ∈ R^n is a path of length n from x to b. (x,b) ∈ R^n holds due to P(n). Therefore, there is a path of length n +1 from a to b. This also implies that (a,b) ∈ R^{n+1}.

➢ Transitive Closures using Paths in Directed Graphs

- ✓ Finding the transitive closure of a relation is equivalent to determining which pairs of vertices in the associated directed graph are connected by a path. With this in mind, we define a new relation.
- ✓ Theorem 1: Let R be a relation on a set A. The connectivity relation R* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.
- ✓ Because R^n consists of the pairs(a, b)such that there is a path of length n from a to b, it follows that R* is the union of all the sets R^n. In other words,

$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

i.e. $R^* = R \cup R^2 \cup R^3 \cup R^4 \cup R^5 \cup \ldots\ldots \cup R^n$.

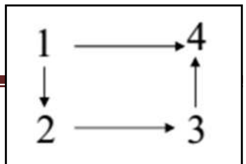
- ✓ Theorem 2: The transitive closure of a relation R equals the connectivity relation R*.
- ✓ Theorem 3: Let M_R be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \cdots \vee M_R^{[n]}$$

➢ **Example 34:** A = {1,2,3,4} R = {(1,2),(1,4),(2,3),(3,4)}. Find its transitive closure

Solution: Let R be a relation on a set A and R* be transitive closure.

$R^* = R \cup R^2 \cup R^3 \cup R^4$



R = {(1,2),(1,4),(2,3),(3,4)}

R^{2}= R.R= [{(1,2),(1,4),(2,3),(3,4)}] .[{(1,2),(1,4),(2,3),(3,4)]

R^2 = {(1,3),(2,4)}

$R^3 = R^2.R$ = [{(1,3),(2,4)}].[{(1,2),(1,4),(2,3),(3,4)}]. R^3 = {(1,4)}

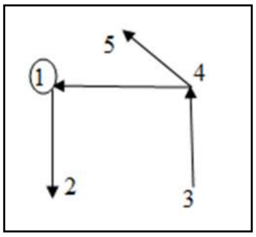
$R^4 = R^3.R$= [{(1,4)}].[{(1,2),(1,4),(2,3),(3,4)] $R^4 = \emptyset$

$R* = R \cup R^2 \cup R^3 \cup R^4$ = {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}.

- **Example 35:** If A={1,2,3,4,5},R={(1,2),(3,4),(4,5),(4,1),(1,1)}.Find its transitive closure.

Solution: Let R be a relation on a set A and R* be transitive closure.

$R* = R \cup R^2 \cup R^3 \cup R^4 \cup R^5$



R = {(1,2),(3,4),(4,5),(4,1),(1,1)}.

R^2 = R.R = {(3,5) (3,1),(4,2),(4,1),(1,2),(1,1)}

$R^3 = R^2.R$ = {(3,2),(3,1),(4,2),(4,1),(1,1),(1,2)}

$R^4 = R^3.R$ = {(3,1),(3,2),(4,1),(4,2),(1,1),(1,2)}

$R^5 = R^4.R$ = {(3,1),(3,2),(4,1),(4,2),(1,1),(1,2)}

R* = {(3,4), (4,5),(3,1),(3,2),(4,1),(4,2),(1,1),(1,2),(3,5)}

- **Example 36:** Find the zero–one matrix of the transitive closure of the relation R where

$$M_R = \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix}$$

Solution: From theorem 3, we have $M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]}$

To find $M_R^{[2]}$ we have, $M_R^{[2]} = M_R \odot M_R = \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix} \odot \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix}$

(1 ∧ 1) ∨ (0 ∧ 0) ∨ (1 ∧ 1)	(1 ∧ 0) ∨ (0 ∧ 1) ∨ (1 ∧ 1)	(1 ∧ 1) ∨ (0 ∧ 0) ∨ (1 ∧ 0)
(0 ∧ 1) ∨ (1 ∧ 0) ∨ (0 ∧ 1)	(0 ∧ 0) ∨ (1 ∧ 1) ∨ (0 ∧ 1)	(0 ∧ 1) ∨ (1 ∧ 0) ∨ (0 ∧ 0)
(1 ∧ 1) ∨ (1 ∧ 0) ∨ (0 ∧ 1)	(1 ∧ 0) ∨ (1 ∧ 1) ∨ (1 ∧ 0)	(1 ∧ 1) ∨ (1 ∧ 0) ∨ (0 ∧ 0)

1 ∨ 0 ∨ 1	0 ∨ 0 ∨ 1	1 ∨ 0 ∨ 0
0 ∨ 0 ∨ 0	0 ∨ 1 ∨ 0	0 ∨ 0 ∨ 0
1 ∨ 0 ∨ 0	0 ∨ 1 ∨ 0	1 ∨ 0 ∨ 0

$$M_R^{[2]} = M_R \odot M_R = \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

To find $M_R^{[3]}$ we have, $M_R^{[3]} = M_R^{[2]} \odot M_R = \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix} \odot \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix}$

$(1 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 1)$	$(1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1)$	$(1 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 0)$
$(0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 1)$	$(0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1)$	$(0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0)$
$(1 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 1)$	$(1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1)$	$(1 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 0)$

$1 \vee 0 \vee 1$	$0 \vee 1 \vee 1$	$1 \vee 0 \vee 0$
$0 \vee 0 \vee 0$	$0 \vee 1 \vee 0$	$0 \vee 0 \vee 0$
$1 \vee 0 \vee 1$	$0 \vee 1 \vee 1$	$1 \vee 0 \vee 0$

$$M_R^{[3]} = \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

$$M_R{}^* = M_R \vee M_R{}^{[2]} \vee M_R{}^{[3]} = \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix} \vee \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix} \vee \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

$$M_R{}^* = \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

- **Transitive Closures using Warshall's Algorithm**
  - ✓ Warshall's algorithm determines whether there is a path between any two nodes in the graph. It does not give the number of the paths between two nodes.
  - ✓ Idea: Compute all paths containing node 1, then all paths containing nodes 1 or 2 or 1 and 2, and so on, until we compute all paths with intermediate nodes selected from the set {1, 2, … n}.
  - ✓ **Warshall's algorithm** is an efficient method of finding the adjacency matrix of the transitive closure of relation R on a finite set S from the adjacency matrix of R. It uses properties of the digraph D, in particular, walks of various lengths in D.
  - ✓ **Warshall's algorithm Step:**
    1. We have |A| = n. Therefore We require W0, W1,W2,W3,…….Wn Warshall sets
       W0 = Relation Matrix of R = M_R.
    2. To find transitive closure of relation R on set A, with |A| = n, compute W_k from W_{k-1} by using following steps:
       a) Copy 1 to all entries in W_k from W_{k-1}, where there is 1 in W_{k-1}.
       b) Find the row numbers R1, R2, R3,….. for which there is 1 in column k in W_{k-1} and column numbers C1, C2, C3,….. for which there is 1 in row k in W_{k-1}.
       c) Mark entries in W_k as 1 for (Ri,Ci). If there are not already 1.
    3. Stop the procedure when W_n is obtained and its gives required transitive closure.

- **Example 37:** Use Warshall Algorithm to find the transitive closures, where A = {1, 2, 3, 4, 5, 6} and R = {(1,3), (2,4), (3,1), (3,5), (4,2), (4,6), (5,3), (6,4)}

  Solution: Step 1: |A| = 6.

  We have to find W0, W1, W2, W3, W4, W5 and W6 Warshall sets.

$$W0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

  Step 2: To find W1 from W0, Consider the first column and first row.

  In R1: 1 is present at C3.

  In C1: 1 is present at R3.

  Thus add new entry in W1 at (R3,C3) =1.

$$W1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

  Step 3: To find W2 from W1, Consider the second column and second row.

  In R2: 1 is present at C4.

  In C2: 1 is present at R4.

  Thus add new entry in W2 at (R4,C4) =1.

$$W2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

  Step 4: To find W3 from W2, Consider the third column and third row.

  In R3: 1 is present at C1, C3, C5.

  In C3: 1 is present at R1, R3, R5.

  Thus add new entry in W3 at (R1,C1), (R1,C3), (R1,C5), (R3,C1), (R3,C3), (R3,C5), (R5,C1), (R5,C3), (R5,C5) =1.

$$W3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 5: To find W4 from W3, Consider the fourth column and fourth row.

In R4: 1 is present at C2, C4, C6.

In C4: 1 is present at R2, R4, R6.

Thus add new entry in W4 at (R2,C2), (R2,C4), (R2,C6), (R4,C2), (R4,C4), (R4,C6), (R6,C2), (R6,C4), (R6,C6) =1.

$$W4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Step 6: To find W5 from W4, Consider the fifth column and fifth row.

In R5: 1 is present at C1, C3, C5.

In C5: 1 is present at R1, R3, R5.

Thus add new entry in W5 at (R1,C1), (R1,C3), (R1,C5), (R3,C1), (R3,C3), (R3,C5), (R5,C1), (R5,C3), (R5,C5) =1.

$$W5 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Step 7: To find W6 from W5, Consider the six column and six row.

In R6: 1 is present at C2, C4, C6.

In C6: 1 is present at R2, R4, R6.

Thus add new entry in W6 at (R2,C2), (R2,C4), (R2,C6), (R4,C2), (R4,C4), (R4,C6), (R6,C2), (R6,C4), (R6,C6) =1.

$$W6 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Hence W6 is the transitive closure

R* = {(1,1), (1,3), (1,5), (2,2), (2,4), (2,6), (3,1), (3,3), (3,5), (4,2), (4,4), (4,6), (5,1), (5,3) (5,5), (6,2), (6,4), (6,6)}

- **Example 38:** Use Warshall Algorithm to find the transitive closures of these relations on {1, 2, 3, 4}: R = {(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)}

Solution: Step 1: |A| = 4.

We have to find W0, W1, W2, W3 and W4 Warshall sets.

R = {(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)}

$$W0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Step 2: To find W1 from W0, Consider the first column and first row.

In R1: 1 is present at C2

In C1: 1 is present at R2, R4.

Thus add new entry in W1 at (R2,C2), (R4,C2) =1

$$W1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Step 3: To find W2 from W1, Consider the second column and second row.

In R2: 1 is present at C1, C2, C3

In C2: 1 is present at R1, R2, R4.

Thus add new entry in W2 at (R1,C1), (R1,C2), (R1,C3), (R2,C1) (R2,C2), (R2,C3), (R4,C1), (R4,C2), (R4,C3) = 1.

$$W2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Step 4: To find W3 from W2, Consider the third column and third row.

In R3: 1 is present at C4

In C3: 1 is present at R1, R2, R4.

Thus add new entry in W3 at (R1,C4), (R2,C4), (R4,C4) = 1

$$W3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Step 5: To find W4 from W3, Consider the fourth column and fourth row.

In R4: 1 is present at C1, C2, C3, C4

In C4: 1 is present at R1, R2, R3, R4.

Thus add new entry in W4 at (R1,C1),(R1,C2),(R1,C3),(R1,C4),(R2,C1),(R2,C2), (R2,C3), (R2,C4), (R3,C1),(R3,C2), (R3,C3), (R3,C4), (R4,C1), (R4,C2), (R4,C3), (R4,C4) = 1.

$$W4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- **Example 39:** Use Warshall Algorithm to find the transitive closures of these relations on A = {1, 2, 3, 4}

  a) R = {(2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)}

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

  b) R = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  c) R = {(1, 1), (1, 4), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 2)}

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- **Example 40:** Find the transitive closure of the relation R on A={1, 2, 3, 4) defined by R = {(1,2), (1,3), (1,4), (2,1), (2,3), (3,4), (3,2), (4,2),(4,3)}.
- **Example 41:** Warshall's algorithm to compute the transitive closure of RUS for the relations R and S defined on A = {1,2,3,4} described as:

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad M_S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

❖ Partition of a Set

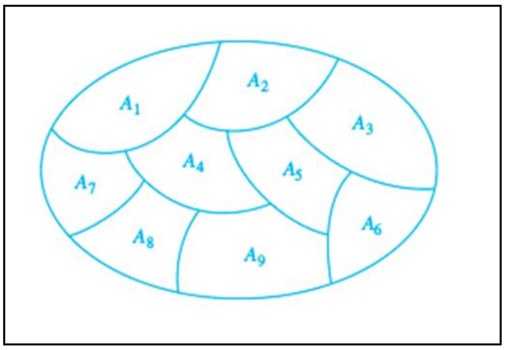
- A partition of a set S is a collection of disjoint nonempty subsets of S that have S as their union.
- In other words, the collection of subsets A_i, i ∈ I (where I is an index set) forms a partition of S if and only if:

  $A_i \neq \emptyset$ for $i \in I$,

  $Ai \cap Aj = \emptyset$ when $i \neq j$,

  And

  $\bigcup_{i \in I} Ai$



- **Example 42:** Suppose that S = {1, 2, 3, 4, 5, 6}. The collection of sets A1 = {1, 2, 3}, A2 = {4, 5}, and A3 = {6} forms a partition of S, because these sets are disjoint and their union is S. i.e. 1) A_1, A_2, A_3 are Non Empty Sets.

  2) $A = A_1 \cup A_2 \cup A_3$ And

  3) $A_1 \cap A_2 = \emptyset$ $A_1 \cap A_3 = \emptyset$ $A_2 \cap A_3 = \emptyset$

  Hence { A_1, A_2, A_3 } form a partition for set A.

❖ Partial Orderings

- A relation R on a set S is called a **partial ordering or partial order** if it is reflexive, antisymmetric, and transitive.
- A set S together with a partial ordering R is called a partially ordered set, or POSET, and is denoted by (S, R).
- Members of S are called elements of the POSET.
- Partial orderings are used to give an order to sets that may not have a natural one.
- Let S = {1,2,3,4,5,6} and R = {(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (6,1), (6,4), (1,4), (6,5), (3,4), (6,2)}. Then R is partial order on S, and (S,R) is a poset.

- **Example 43:** Show that the "greater than or equal" relation (≥) is a partial ordering on the set of integers.

Solution:

- ✓ Because a ≥ a for every integer a, ≥ is reflexive.
- ✓ If a ≥ b and b ≥ a, then a = b. Hence, ≥ is antisymmetric.
- ✓ Finally, ≥ is transitive because a ≥ b and b ≥ c imply that a ≥ c.
- ✓ It follows that ≥ is a partial ordering on the set of integers and (Z, ≥) is a poset.

- **Example 44:** Is the "divides" relation on the set of positive integers reflexive, symmetric and transitive?

  Solution:

  - ✓ Because a | a whenever a is a positive integer, the "divides" relation is reflexive. (Note that if we replace the set of positive integers with the set of all integers the relation is not reflexive because by definition 0 does not divide 0.)
  - ✓ This relation is not symmetric because 1| 2, but 2 ∤ 1. It is antisymmetric, for if a and b are positive integers with a | b and b | a, then a = b.
  - ✓ Suppose that a divides b and b divides c. Then there are positive integers k and l such that b = ak and c = bl. Hence, c = a(kl), so a divides c. It follows that this relation is transitive.

- **Example 45:** The divisibility relation | is a partial ordering on the set of positive integers, because it is reflexive, antisymmetric, and transitive, as shown in above example. We see that (Z+, |) is a poset. Recall that (Z+ denotes the set of positive integers.)

- **Example 46:** Show that the inclusion relation ⊆ is a partial ordering on the power set of a set S.

  Solution: Because A ⊆ A whenever A is a subset of S,⊆ is reflexive. It is antisymmetric because A ⊆ B and B ⊆ A imply that A = B. Finally, ⊆ is transitive, because A ⊆ B and B ⊆ C imply that A ⊆ C. Hence, ⊆ is a partial ordering on P(S), and (P (S),⊆) is a poset.

- **Example 47:** Let R be the relation on the set of people such that xRy if x and y are people and x is older than y. Show that R is not a partial ordering.

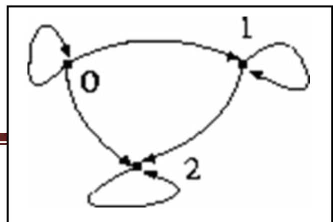
  Solution: Relation R is antisymmetric because if a person x is older than a person y, then y is not older than x. That is, if (x, y) ∈ R, then (y, x) ∉ R.

  The relation R is transitive because if person x is older than person y and y is older than person z, then x is older than z. That is, if xRy and yRz, then xRz.

  Relation R is not reflexive, because no person is older than himself or herself. i.e (x, x) ∉ R for all people x. It follows that R is not a partial ordering.

- **Example 48:** Let A={0,1,2}and R={(0,0),(0,1),(0,2),(1,1),(1,2),(2,2)}.Show R is a partial order relation.

  Solution: The digraph for R on the right implies



Reflexive: Loops on every vertex.

Antisymmetric: No arrows of type (a,b) and (b,a).

Transitive:(0,1), (1,2) also we have (0,2).

- **Partial Orderings Notation**
  - ✓ A set S together with a partial ordering R is called a partially ordered set, or POSET, and is denoted by (S, R).
  - ✓ In different posets different symbols such as $\leq$, $\subseteq$, and $|$, are used for a partial ordering.
  - ✓ Customarily, the notation $\mathbf{a \preceq b}$ is used to denote that $(a, b) \in R$ in an arbitrary poset (S,R).
  - ✓ This notation is used because the "less than or equal to" relation on the set of real numbers is the most familiar example of a partial ordering and the symbol $\preceq$ is similar to the $\leq$ symbol.
  - ✓ Note that the symbol $\preceq$ is used to denote the relation in **any poset,** not just the "less than or equals" relation.
  - ✓ The notation $a \prec b$ denotes that $a \preceq b$, but $a \neq b$. Also, we say "*a* is less than *b*" or "*b* is greater than *a*" if $a \prec b$.
- Suppose that $\preceq$ is a partial order on a set S. The relations $\succeq$, $\prec$, $\succ$, $\perp$, and $\parallel$ are defined as follows:
  - ✓ $x \succeq y$ if and only if $y \preceq x$
  - ✓ $x \prec y$ if and only if $x \preceq y$ and $x \neq y$
  - ✓ $x \succ y$ if and only if $y \prec x$
  - ✓ $x \perp y$ if and only if $x \preceq y$ or $y \preceq x$
  - ✓ $x \mid\mid y$ if and only if neither $x \preceq y$ nor $y \preceq x$.
- Note that $\succeq$ is the inverse of $\preceq$, and $\succ$ is the inverse of $\prec$. Note also that $x \preceq y$ if and only if either $x \prec y$ or $x = y$, so the relation $\prec$ completely determines the relation $\preceq$. Finally, note that $x \perp y$ means that x and y are related in the partial order, while $x \parallel y$ means that x and y are unrelated in the partial order. Thus, the relations $\perp$ and $\mid\mid$ are complements of each other, as sets of ordered pairs.

- **Basic Properties**
  - ✓ The inverse of a partial order is also a partial order.

    Proof: Clearly the reflexive, antisymmetric and transitive properties hold for $\succeq$.
  - ✓ If $\preceq$ is a partial order on S and A is a subset of S, then the restriction of $\preceq$ to A is a partial order on A

Proof: The reflexive, antisymmetric, and transitive properties given above hold for all x, y, z ∈ S and hence hold for all x, y, z ∈ A.

- **Comparable Element**
  - The elements a and b of a poset (S,⪯) are called comparable if either a⪯b or b⪯a. When a and b are elements of S such that neither a⪯b nor b⪯a, a and b are called incomparable.
  - In the poset (Z^+,|), are the integers 3, 9 and 5, 7 comparable?
  - The integers 3 and 9 are comparable, because 3|9. The integers 5 and 7 are incomparable, because 5 ∤ 7 and 7 ∤ 5.
  - The adjective "partial" is used to describe partial orderings because pairs of elements may be incomparable. When every two elements in the set are comparable, the relation is called a total ordering.
- **Total Ordered Set**
  - If (S,⪯) is a poset and every two elements of S are comparable, S is called a totally ordered Or linearly ordered set, and ⪯ is called a total order or a linear order. A totally ordered set is also called a chain.
  - The poset (Z, ≤) is totally ordered, because a ≤ b or b ≤ a whenever a and b are integers.
  - The poset (Z^+,|) is not totally ordered because it contains elements that are incomparable, such as 5 and 7.Therefore Poset (Z, ≤) is a chain and (Z^+,|) is not a chain.
- **Well-Ordered Set**
  - (S,⪯) is a well-ordered set if it is a poset such that ⪯ is a total ordering and every nonempty subset of S has a least element.

  - **Example 49:** Which of these relations on {0, 1, 2, 3} are partial orderings? Determine the properties of a partial ordering that the others lack.

    a) {(0, 0), (1, 1), (2, 2), (3, 3)}

    b) {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}

    c) {(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)}

    d) {(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}

    e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0),(2, 2), (3, 3)}

    Solution:

    a) Clearly this relation is reflexive because each of 0, 1, 2, and 3 is related to itself. The relation is antisymmetric, because the only way for a to be related to b is for a = b. Similarly, the relation is transitive, because if a is related to b, and b is related to c, then necessarily a= b = c so a is related to c (because the relation is reflexive).

b) This is not a partial ordering, because although the relation is reflexive, it is not antisymmetric (we have 2R3 and 3R2, but 2 ≠ 3), &not transitive (3R2 and 2R0, but 3 is not related to 0).

c) This is a partial ordering, because it is clearly reflexive; is antisymmetric (we just need to note that (1,2) is the only pair in the relation with unequal components); and is transitive (for the same reason).

d) This is a partial ordering because it is the "less than or equal to" relation on {1, 2, 3} together with the isolated point 0.

e) This is not a partial ordering. The relation is clearly reflexive, but it is not antisymmetric (0R1 and 1R0, but 0≠1) and not transitive (2R0 and 0R1, but 2 is not related to 1).

- **Example 50:** Which of these are posets? a) (Z, =) b) (Z, ≠) c) (Z, ≥) d) (Z, $\not>$)

a) This is a poset. The only ordered pairs we will have in this relation is (a, a) for all a ∈ Z. This would mean that the relation is reflexive, antisymmetric, and transitive.

b) This is not a poset because it is not reflexive. We cannot have the order pair (a, a) for all a ∈ Z. This relation is also not antisymmetric and not transitive.

c) This is a poset. For reflexive, we can have the ordered pair (a, a) for all a ∈ Z. This is also antisymmetric because consider the ordered pair (a, b) and a = b, this would mean that a > b. If this is the case, then b>a is not true and you cannot have (b, a). This is also transitive because if a > b, b>c, and a =b=c. Then it follows that a>c for all a, b, c ∈ Z.

d) This is not a poset because it is not reflexive. Consider 2 $\not>$ 2, since this is not true, we cannot have (2, 2). This relation is also not antisymmetric and not transitive.

❖ Hasse Diagram

- A Hasse Diagram is a type of mathematical diagram used to represent a finite partially ordered set, in the form of a drawing of its transitive reduction. A visual representation of a partial ordering.
- Procedure to Draw Hasse Diagram:

1. Start with the directed graph for this relation. Because a partial ordering is reflexive, a loop (a, a) is present at every vertex a. Remove these loops.
2. Next, remove all edges that must be in the partial ordering because of the presence of other edges and transitivity. That is, remove all edges(x,y) for which there is an element z ∈ S such that x≺ z and z ≺ x.
3. Finally, arrange each edge so that its initial vertex is below its terminal vertex.
4. Remove all the arrows on the directed edges, because all edges point “upward” toward their terminal vertex.

- **Example 51:** Consider the directed graph for the partial ordering {(a, b) | a ≤ b} on the set {1,2,3,4}, shown in Figure (a) and draw Hasse Diagram for it.

  Solution:

  Because this relation is a partial ordering, it is reflexive, and its directed graph has loops at all vertices. Consequently, we do not have to show these loops because they must be present; in Figure (b) loops are not shown.

  Because a partial ordering is transitive, we do not have to show those edges that must be present because of transitivity. For example, in Figure (c) the edges (1,3), (1,4), and (2,4) are not shown because they must be present.

  If we assume that all edges are pointed "upward" (as they are drawn in the figure), we do not have to show the directions of the edges; Figure (c) does not show directions.

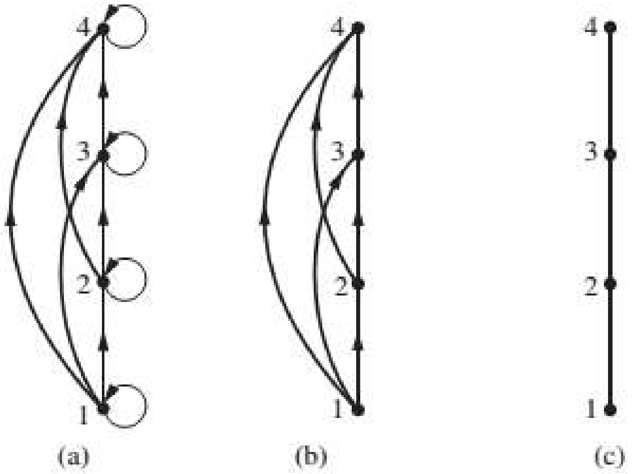


Fig: Constructing the Hasse Diagram for ({1,2,3,4},≤).

- **Example 52:** Draw the Hasse diagram representing the partial ordering {(a, b)|a divides b} on {1, 2, 3, 4, 6, 8, 12}.

  Solution:

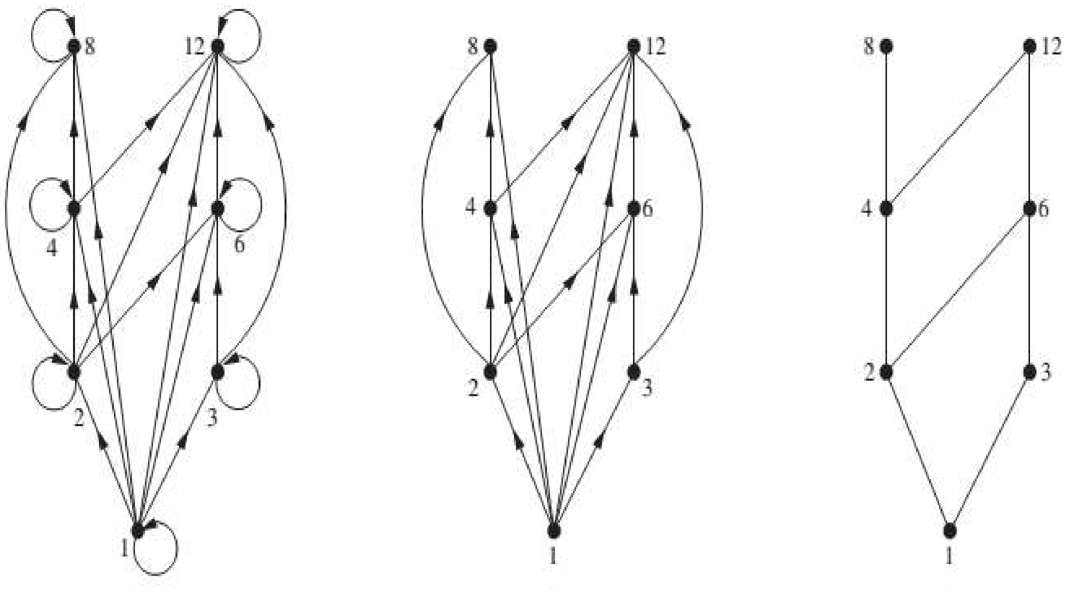
  R={(1,1),(1,2),(1,3),(1,4),(1,6),(1,8),(1,12),(2,2),(2,4),(2,6),(2,8),(2,12),(3,3),(3,6),(3,12),(4,4),(4,8),(4,12),(6,6),(6,12),(8,8),((12,12)}.

  Begin with the digraph for this partial order, as shown in Fig (a). Remove all loops, as shown in Fig (b).

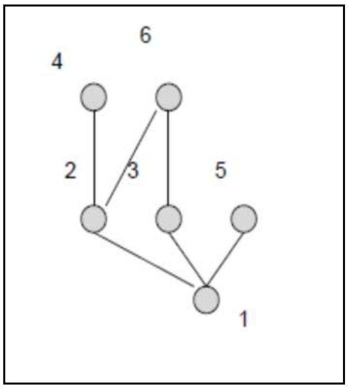
  Then delete all the edges implied by the transitive property. These are (1,4), (1,6), (1,8), (1,12), (2,8), (2,12), and (3,12).

  Arrange all edges to point upward, and delete all arrows to obtain the Hasse diagram.

  The resulting Hasse diagram is shown in Fig (c).



- **Example 53:** Draw the Hasse diagram for divisibility on the set (Homework for Students)
  - a) {1, 2, 3, 4, 5, 6}
  - b) {3, 5, 7, 11, 13, 16, 17}
  - c) {2, 3, 5, 10, 11, 15, 25}
  - d) {1, 3, 9, 27, 81, 243}



- **Example 54:** Draw the Hasse diagram for divisibility on the set
  - a) {1, 2, 3, 4, 5, 6, 7, 8}
  - b) {1, 2, 3, 5, 7, 11, 13}
  - c) {1, 2, 3, 6, 12, 24, 36, 48}
  - d) {1, 2, 4, 8, 16, 32, 64}
  - e) {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}

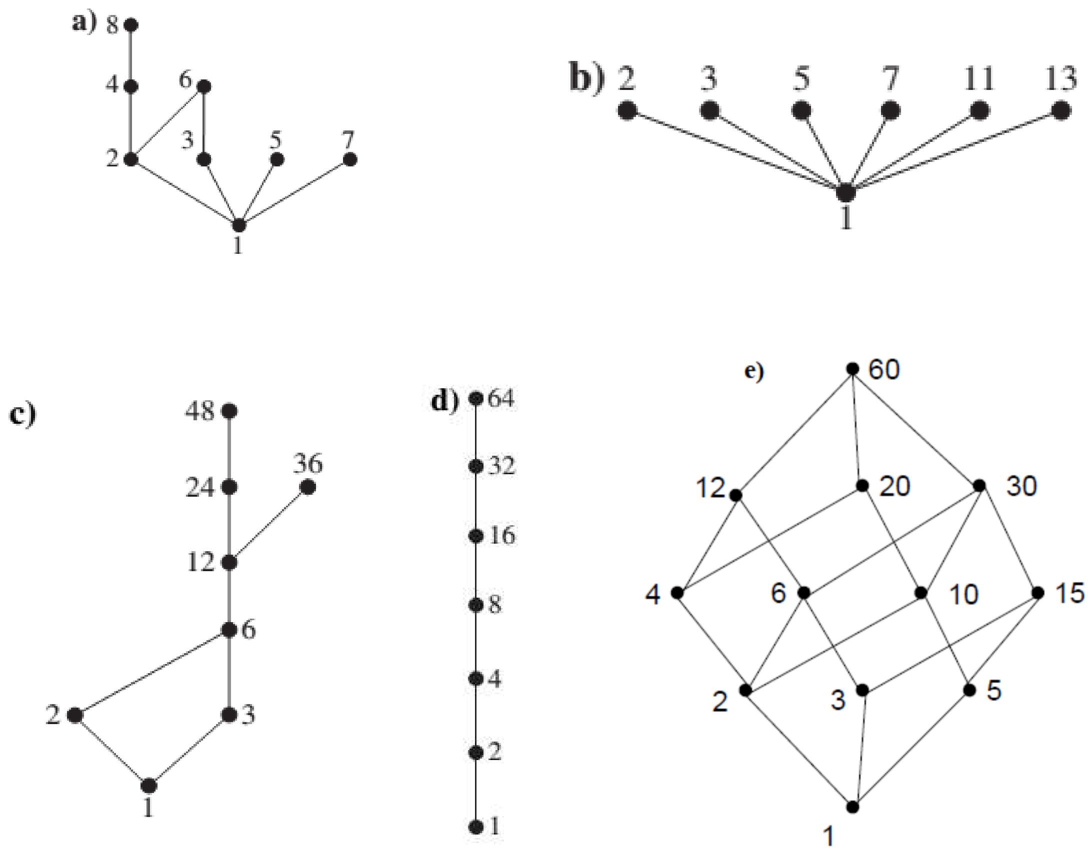
  Solution:

  We put x above y if y divides x.

  We draw a line between x and y, where y divides x, if there is no number z in our set, other than x or y, such that $y \mid z \wedge z \mid x$.

  Note that in part (b) the numbers other than 1 are all (relatively) prime, so the Hasse diagram is short and wide,

  Whereas in part (d) the numbers all divide one another, so the Hasse diagram is tall and narrow.



❖ **Chain and Anti Chain**

- A **chain** in a poset P is a subset C ⊆ P such that any two elements in C are comparable.
- An **antichain** in a poset P is a subset A ⊆ P Such that no two elements in A are comparable.

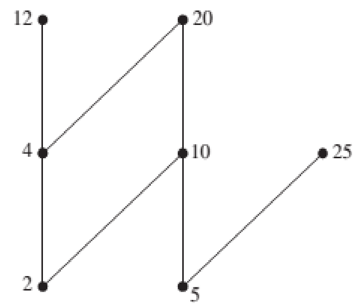
❖ **Elements of POSETS**

- **Maximal Elements:**
  - ✓ Let (A,⪯) be a poset. Then a ∈ A is maximal in the poset if there is no element b ∈ A such that a ≺ b.
- **Minimal Elements:**
  - ✓ Let (A,⪯) be a poset. Then a ∈ A is minimal in the poset if there is no element b ∈ A such that b ≺ a.
- **Maximal and Minimal elements** are easy to spot using a Hasse diagram. They are the “top” and “bottom” elements in the diagram. There can be more than one minimal and maximal element in a poset.
- **Greatest Element:**
  - ✓ Let (A,⪯) be a poset. Then a ∈ A is the greatest element if for every element b ∈ A, b⪯a.

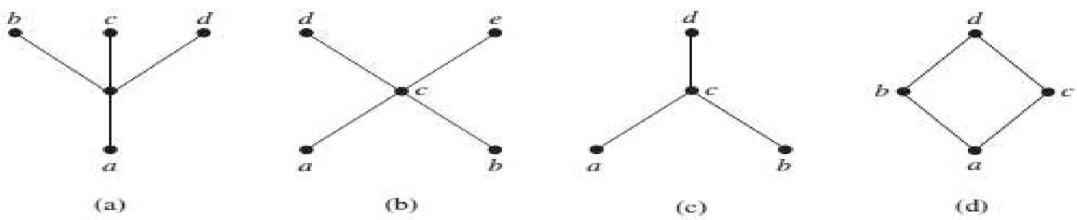
- **Least Element:**
  - ✓ Let (A,≤) be a poset. Then a ∈ A is the least element if for every element b ∈ A, a≤b.
- **Upper Bound:**
  - ✓ Let S ⊆ A in the poset (A,≤). If there exists an element *u* ∈ A such that s ≤ *u* for all s ∈ S, then *u* is called an upper bound of S.
- **Lower Bound:**
  - ✓ Let S ⊆ A in the poset (A,≤). If there exists an element *l* ∈ A such that *l* ≤ s for all s ∈ S, then *l* is called a lower bound of S.
- **Least Upper Bound:**
  - ✓ If a is an upper bound of S such that *a* ≤ *u* for all upper bound u of S then *a* is the least upper bound of S, denoted by lub(S).
- **Greater Lower Bound:**
  - ✓ If a is a lower bound of S such that *l* ≤ *a* for all lower bound *l* of S then *a* is the greatest lower bound of S, denoted by glb(S).

- **Example 55:** Which elements of the poset ({2, 4, 5, 10, 12, 20, 25},|) are maximal, and which are minimal?

  Solution: The Hasse diagram in figure for this poset shows that the maximal elements are 12, 20, and 25, and the minimal elements are 2 and 5. As this example shows, a poset can have more than one maximal element and more than one minimal element.



- **Example 56:** Determine whether the posets represented by each of the Hasse diagrams in below figure have a greatest element and a least element.



Solution: The least element of the poset with Hasse diagram (a) is a. This poset has no greatest element.

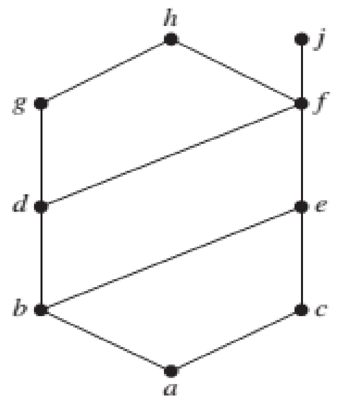
The poset with Hasse diagram (b) has neither a least nor a greatest element.

The poset with Hasse diagram (c) has no least element. Its greatest element is d.

The poset with Hasse diagram (d) has least element a and greatest element d.

- **Example 57:** Find the lower and upper bounds of the subsets {a, b, c}, {j, h}, and {a, c, d, f} in the poset with the Hasse diagram shown in figure below.

  Solution: The upper bounds of {a, b, c} are e, f, j, and h, and its only lower bound is a. There are no upper bounds of {j, h}, and its lower bounds are a, b, c, d, e, and f. The upper bounds of {a, c, d, f} are f, h, and j, and its lower bound is a.

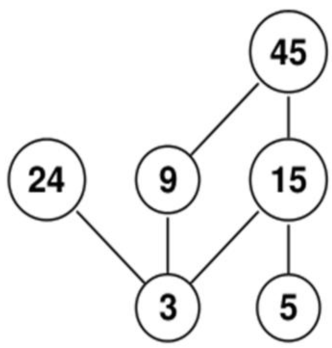


- **Example 58:** Find the glb and the lub of {b, d, g}, if they exist, of Example 57.

  Solution: The upper bounds of {b, d, g} are g and h. Because $g \prec h$, g is the lub. The lower bounds of {b, d, g} are a and b. Because $a \prec b$, b is the glb.

- **Example 59:** Answer these questions for the poset ({3, 5, 9, 15, 24, 45}, |).

  a. Find the maximal elements.
  b. Find the minimal elements.
  c. Is there a greatest element?
  d. Is there a least element?
  e. Find all upper bounds of {3, 5}.
  f. Find the least upper bound of {3, 5}, if it exists.
  g. Find all lower bounds of {15, 45}.
  h. Find the greatest lower bound of {15, 45}, if it exists.



  Solution:

  a. Maximal elements are those that do not divide any other elements of the set. In this case 24 and 45 are the only numbers that meet that requirement.
  b. Minimal elements are those that are not divisible by any other elements of the set. In this case 3 and 5 are the only numbers that meet that requirement.
  c. There is no greatest element because this element would have to be a number that all other elements divide. Since our maximal elements are 24 and 45, and they do not divide each other, we do not have a greatest element.

d. There is no least element because this element would be a number that can divide all other elements. Since our minimal elements are 3 and 5, and they do not divide each other, we do not have a least element.

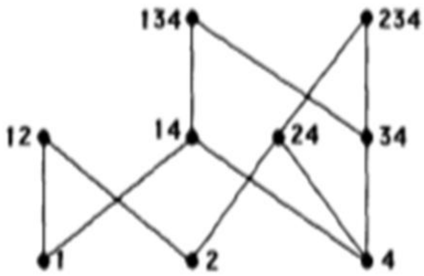
e. We want to find all elements that both 3 and 5 divide. Clearly only 15 and 45 meet this requirement.

f. The least upper bound is 15 since it divides 45 (see part (e)).

g. We want to find all elements that divide both 15 and 45. Clearly only 3, 5, and 15 meet this requirement.

h. The number 15 is the greatest lower bound, since both 3 and 5 divide it (see part (g)).

- **Example 60:** Answer the questions for the poset ({{1}, {2}, {4}, {1, 2}, {1, 4}, {2, 4}, {3, 4}, {1, 3, 4}, {2, 3, 4}}, ⊆).



a. Find the maximal elements

b. Find the minimal elements.

c. Is there a greatest element?

d. Is there a least element?

e. Find all upper bounds of {{2}, {4}}.

f. Find the least upper bound of {{2}, {4}}, if it exists.

g. Find all lower bounds of {{1, 3, 4}, {2, 3, 4}}.

h. Find the greatest lower bound of {{1, 3, 4}, {2, 3, 4}},if it exists.

Solution:

a. The maximal elements are the ones without any elements lying above them in the Hasse diagram, namely {1,2}, {1,3,4}, and {2,3,4}.

b. The minimal elements are the ones without any elements lying below them in the Hasse diagram, namely { 1} , { 2} , and { 4} .

c. There is no greatest element, since there is more than one maximal element, none of which is greater than the others.

d. There is no least element, since there is more than one minimal element, none of which is less than the others.

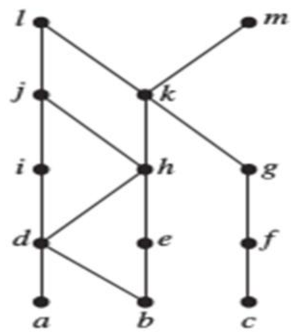
e. The upper bounds are the sets containing both {2} and {4} as subsets, i.e., the sets containing both 2 & 4 as elements. Pictorially, these are the elements lying above both {2}&{4} (in the sense of there being a path in the diagram), namely {2,4} & {2,3,4}.

f. The least upper bound is an upper bound that is less than every other upper bound. We found the upper bounds in part (e), and since {2,4} is less than (i.e., a subset of) {2,3,4}, we conclude that {2,4} is the least upper bound.

g. To be a lower bound of {l, 3, 4} and {2, 3, 4}, a set must be a subset of each, and so must be a subset of their intersection, {3, 4}. There are only two such subsets in our poset, namely {3, 4} and {4}. In the diagram, these are the points which lie below (in the path sense) both {1, 3, 4} and {2, 3, 4}.

h. The greatest lower bound is a lower bound that is greater than every other lower bound. We found the lower bounds in part (g), and since {3,4} is greater than (i.e., a superset of) {4}, we conclude that {3,4} is the greatest lower bound.

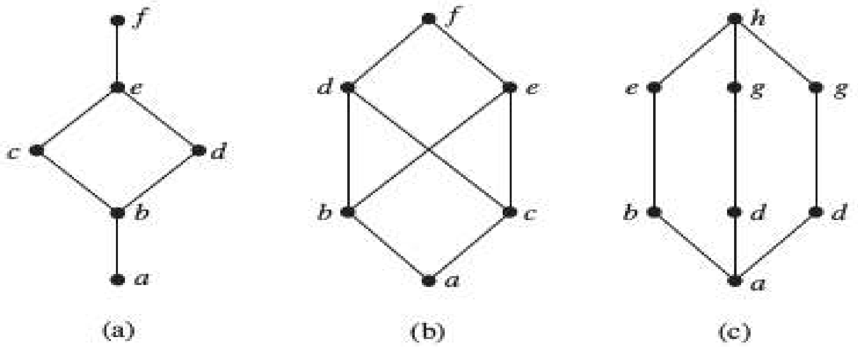
- **Example 61:** Find the following for given poset diagram(HW)



a. Maximal elements, Minimal elements.
b. Is there a greatest element
c. Is there a least element?
d. Find all upper bounds of {a, b, c}.
e. Find the least upper bound of {a, b, c}, if it exists.
f. Find all lower bounds of {f, g, h}.
g. Find the greatest lower bound of {f, g, h}, if it exists.

❖ Lattices

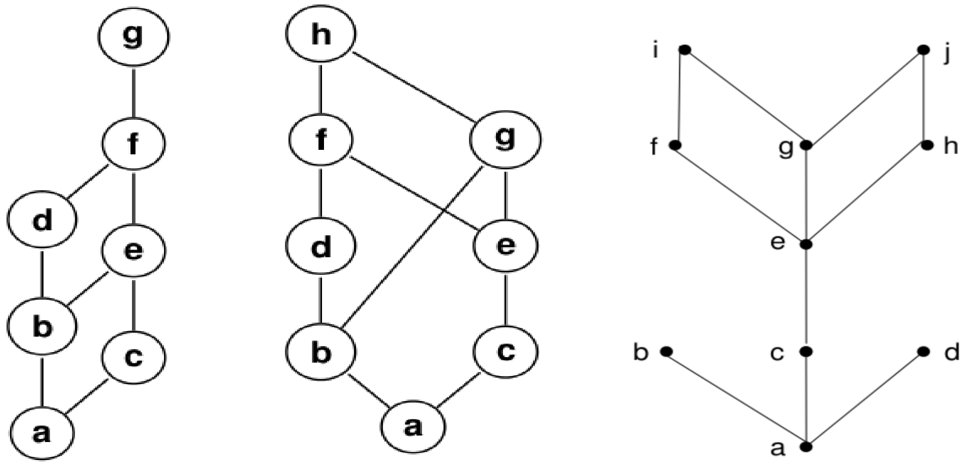
- A poset in which every pair of elements has both a least upper bound and a greatest lower bound is called a lattice.
- A lattice A is called a complete lattice if every subset S of A admits a glb and a lub in A.
- To show that a partial order is not a lattice, it suffices to find a pair that does not have an lub or a glb (i.e., a counter-example)
- For a pair not to have an lub/glb, the elements of the pair must first be incomparable.
- You can then view the upper/lower bounds on a pair as a sub-Hasse diagram: If there is no maximum/minimum element in this sub-diagram, then it is not a lattice.
- Lattices have many special properties. Furthermore, lattices are used in many different applications such as models of information flow and play an important role in Boolean algebra.
- **Example 62:** Determine whether the posets with these Hasse diagrams are lattices.



Solution:

Fig a and c are lattices. Fig b is not because, {b,c} has no lub However, it has a glb={a}.

- **Example 63:** Determine whether the posets with these Hasse diagrams are lattices.



(a) (b) c)

Solution:

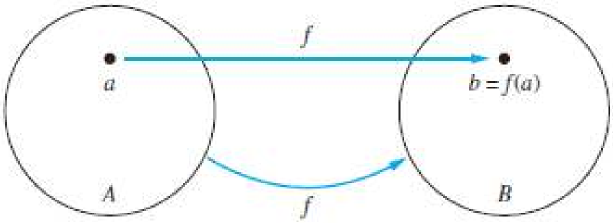
a) Yes. Every two elements will have a least upper bound and greatest lower bound.

b) No. If we take the elements b and c, then we will have f, g, and h as the upper bound, but none of them will be the least upper bound.

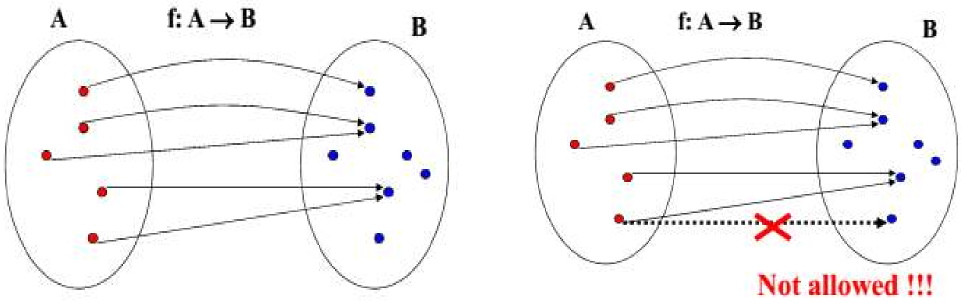
c) No, because the pair {b,c}does not have a least upper bound.

❖ Function-Introduction

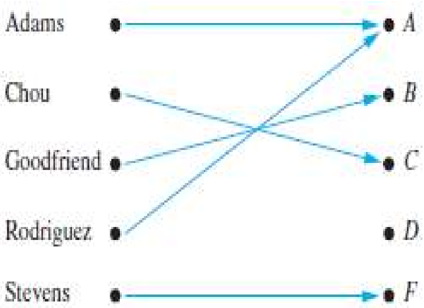
- **Definition:** Let A and B be nonempty sets. A function ***f*** from A to B, denoted ***f*: A →B**, is an assignment of exactly one element of B to each element of A.
- We write ***f (a) = b*** if *b* is the unique element of *B* assigned by the function *f* to the element *a* of *A*.
- If f is a function from A to B, we say that A is the domain of f and B is the codomain of f.
- If f (a) = b, we say that b is the image of a and a is a preimage of b.
- The range, or image, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f maps A to B.



- When we define a function we specify its domain, its codomain, and the mapping of elements
  of the domain to elements in the codomain.
- Two functions are equal when they have the same domain, have the same codomain, and map each element of their common domain to the same element in their common codomain.
- Note that if we change either the domain or the codomain of a function, then we obtain a different function. If we change the mapping of elements, then we also obtain a different function.



- **Example 64:** Suppose that each student in a discrete mathematics class is assigned a letter grade from the set {A,B,C,D, F}. And suppose that the grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens. What are the domain, codomain, and range of the function that assigns grades to students?



Solution: Let G be the function that assigns a grade to a student in our discrete mathematics class. Note that G(Adams) = A, for instance.

The domain of G is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}, and

The codomain is the set {A,B,C,D, F}.

The range of G is the set {A,B,C, F}, because each grade except D is assigned to some student.

- **Example 65:** Let R be the relation with ordered pairs (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22). Here each pair consists of a graduate student and this student's age. Specify a function determined by this relation.

  Solution: If f is a function specified by R, then f (Abdul) = 22, f (Brenda) = 24,f (Carla) = 21, f (Desire) = 22, f (Eddie) = 24, and f (Felicia) = 22.

  (Here, f (x) is the age of x, where x is a student.) For the domain, we take the set {Abdul, Brenda, Carla, Desire, Eddie, Felicia}.

  We also need to specify a codomain, which needs to contain all possible ages of students. Because it is highly likely that all students are less than 100 years old, we can take the set of positive integers less than 100 as the codomain.

  The range of the function we have specified is the set of different ages of these students, which is the set {21, 22, 24}.

- **Example 66:** Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example, f (11010) = 10. Then, the domain of f is the set of all bit strings of length 2 or greater, and both the codomain and range are the set {00, 01, 10, 11}.

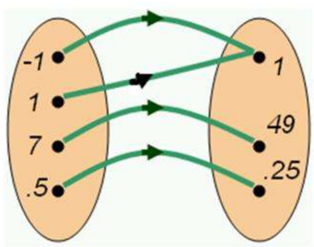
- **Example 67:** Let f : Z → Z assign the square of an integer to this integer. Then, f (x) = x2, where the domain of f is the set of all integers, the codomain of f is the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, {0, 1, 4, 9, . . . }.

- **Example 68:** If we write (define) a function as: $f(x)=x^2$ then we say: **'*f* of x equals x squared'** and we have:

  $f(-1) = 1$ $\quad$ $f(1) = 1$ $\quad$ $f(2) = 4$

  $f(5) = 25$ $\quad$ $f(7) = 49$ $\quad$ and so on.

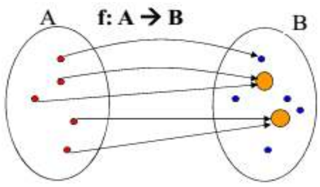
  This function *f* maps numbers to their squares.

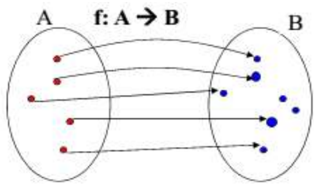


❖ Type of Function::

1. Injective / One-to-one function

✓ A function *f* is said to be one-to-one, or injective, if and only if *f*(x) = *f*(y) implies x = y for all x, y in the domain of *f*. A function is said to be an injection if it is one-to-one.



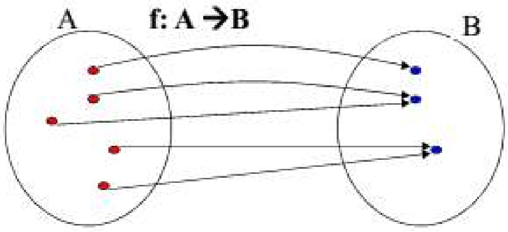


Not injective function **Injective function**

- ✓ Note that a function f is one-to-one if and only if $f(x) \neq f(y)$ whenever $x \neq y$. This way of expressing that f is one-to-one is obtained by taking the contrapositive of the implication in the definition.

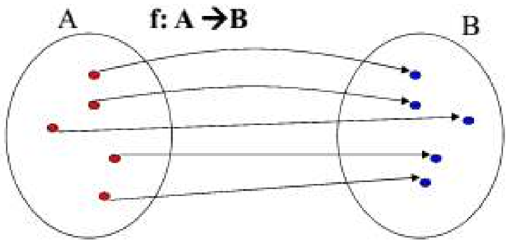
2. Surjective / Onto function

- ✓ A function f from A to B is called onto, or surjective, if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$.



3. Bijective / One-to-one Correspondent

- ✓ A function f is called a bijection if it is both one-to one (injection) and onto (surjection).



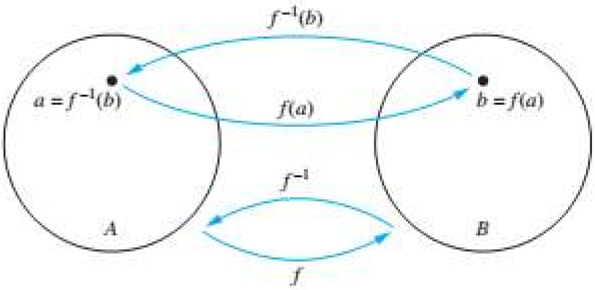
4. Identity function

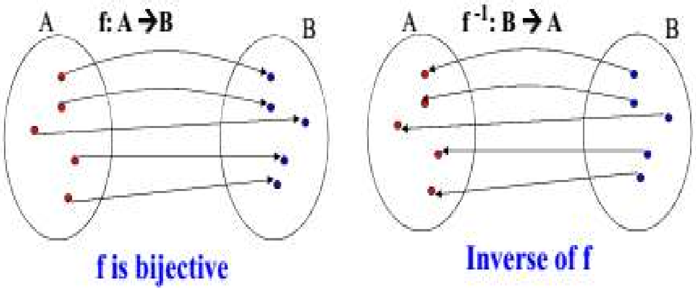
- ✓ Let A be a set. The identity function on A is the function $i_A: A \rightarrow A$ where $i_A(x) = x$.
- ✓ Example: Let A = {1, 2, 3} Then:

  $i_A(1) = 1$ $i_A(2) = 2$ and $i_A(3) = 3$.

5. Inverse functions

- ✓ Let f be a bijection from set A to set B. The inverse function of f is the function that assigns to an element b from B the unique element a in A such that $f(a) = b$.
- ✓ The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$, when $f(a) = b$.
- ✓ If the inverse function of f exists, f is called invertible.



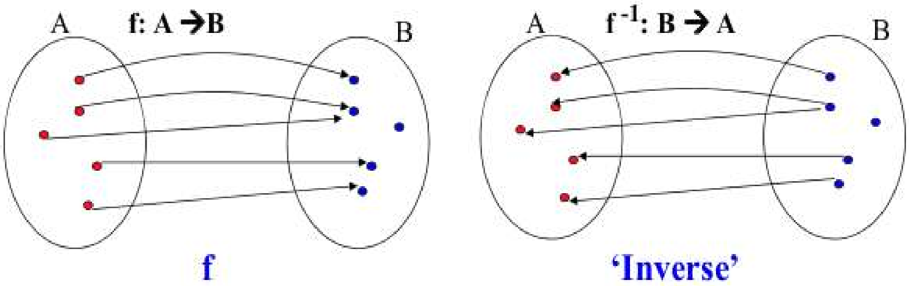


- ✓ Note: If *f* is not a bijection then it is not possible to define the inverse function of *f*.

  Solution:

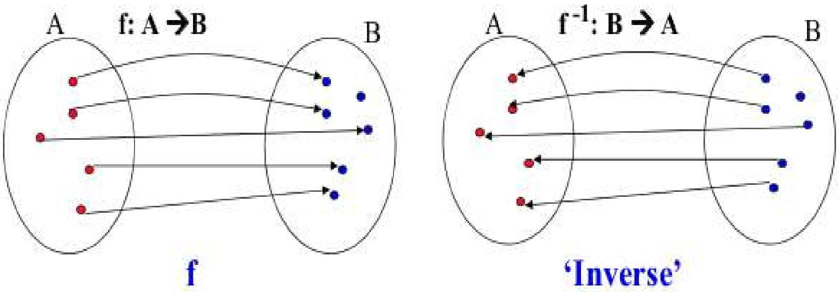
  Case 1: Assume *f* is not one-to-one:

  Inverse is not a function. One element of B is mapped to two different elements.



Case 2: Assume *f* is not onto:

Inverse is not a function. One element of B is not assigned any value in B.

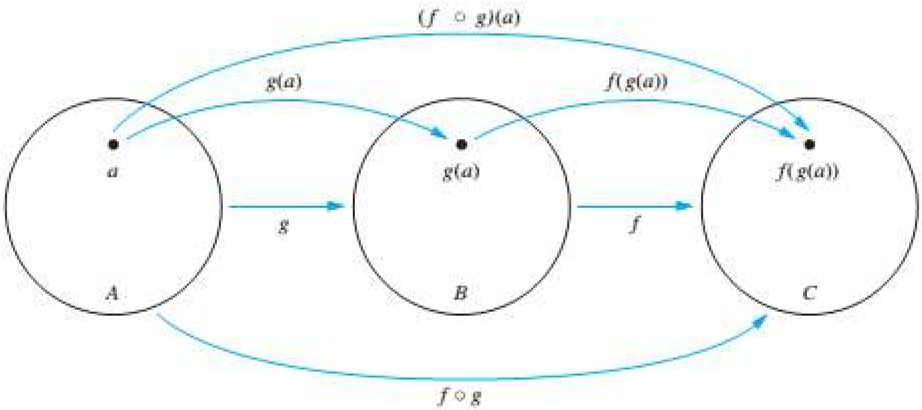


6. Composition of Functions

Let ***g*** be a function from the set A to the set B and let ***f*** be a function from the set B to the set C. The composition of the functions **f and g**, denoted for all a ∈ A by **f ∘ g**, is defined by

$$(f \circ g)(a) = f(g(a)).$$

In other words, $f \circ g$ is the function that assigns to the element a of A the element assigned by f to $g(a)$. That is, to find $(f \circ g)(a)$ we first apply the function g to a to obtain $g(a)$ and then we apply the function f to the result $g(a)$ to obtain $(f \circ g)(a) = f(g(a))$. Note that the composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f. In figure below the composition of functions is shown.

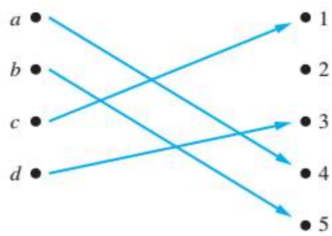


- **Example 69:** Let A = {1,2,3} and B = {a, b, c, d} Find (f O g)

g : A →A	f: A →B	Solution:
1 → 3	1 → b	f O g : A →B:
2 → 1	2 → a	1 → d
3 → 2	3 → d	2 → b
		3 → a

- **Example 70:** Determine whether the function f from{a, b, c, d} to {1,2,3,4,5} with f(a)=4, f(b)=5, f(c)=1, and f(d)=3 is one-to-one.

  Solution: The function f is one-to-one because f takes on different values at the four elements of its domain.

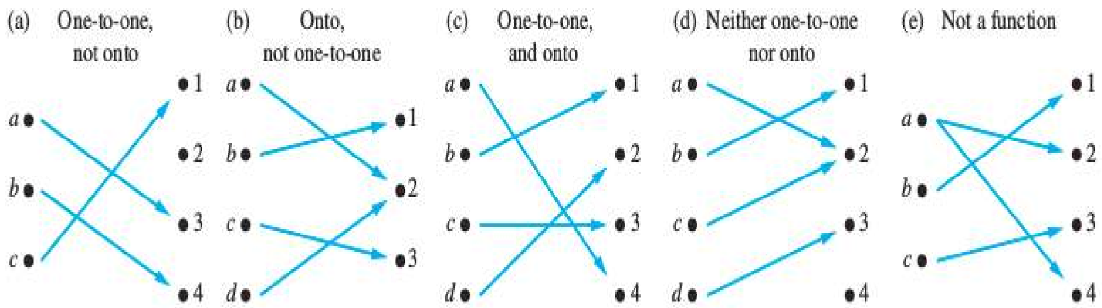


- **Example 71:** Determine whether the function f(x)= x^2 from the set of integers to the set of integers is one-to-one.

  Solution: The function f(x)= x^2 is not one-to-one because, for instance, f(1)= f(−1) =1, but $1 \neq -1$.

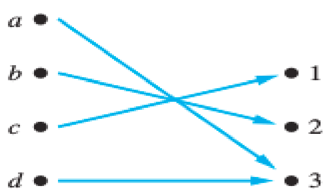
  Note that the function (x)= x^2 with its domain restricted to Z^+ is one-to-one.

- **Example 72:** Identify the following Function.



- **Example 73:** Let f be the function from {a, b, c, d} to {1, 2, 3} defined by f(a)= 3, f(b)= 2, f(c)= 1, and f(d)= 3. Is *f* an onto function?

  Solution: Because all three elements of the codomain are images of elements in the domain, we see that f is onto. Note that if the codomain were {1, 2, 3, 4}, then *f* would not be onto.



- **Example 74:** Let f be the function from {a, b, c} to {1, 2, 3} such that f(a) = 2, f(b) = 3, and f(c) = 1. Is f invertible, and if it is, what is its inverse?

  Solution: The function *f* is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by *f*, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.
- **Example 75:** Let g be the function from the set {a, b, c} to itself such that g(a) = b, g(b) = c, & g(c) = a Let f be the function from the set {a, b, c} to the set {1, 2, 3} such that f(a) = 3, f(b) = 2, and f(c) = 1. What is the composition of (fOg)?

  Solution: The composition f O g is defined by

  (fOg)(a) = f(g(a)) = f(b) = 2,

  (fOg)(b) = f(g(b)) = f(c) = 1, and

  (fOg)(c) = f(g(c)) = f(a) = 3.
- **Example 76:** Let f and g be the functions from the set of integers to the set of integers defined by f (x) = 2x + 3 and g(x) = 3x + 2. What is the composition of (f Og)? What is the composition of (gOf)?

  Solution**:** Both the compositions (fOg) and (gOf) are defined. Moreover,

  (fOg)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7

  And (gOf)(x) = g(f (x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11
- **Example 77:** Let f(x)= x+2, g(x)= x-2 and h(x)=3x for x $\in$ R, where R is set of real numbers. Find (gOf), (fOg), (fOf), (gOg), (fOh), (hOg), (hOf),(fOhOg)

  Solution:

(gOf) = g(f(x)) = g(x+2) = (x+2)-2 = x

(fOg) = f(g(x)) = f(x-2) = (x-2)+2 = x

(fOf) = f(f(x)) = f(x+2) = (x+2)+2 = x+4

(gOg) = g(g(x)) = g(x-2) = (x-2)-2 = x-4

(fOh) = f(h(x)) = f(3x) = (3x)+2 = 3x+2

(hOg) = h(g(x)) = h(x-2) =3(x-2) = 3x-6

(hOf) = h(f(x)) = h(x+2) =3(x+2) = 3x+6

(fOhOg) = fOh(g(x)) = fOh(x-2) =f(h(x-2)) = f(3x-6) = (3x-6)+2 = 3x-4

❖ **Important Note:**

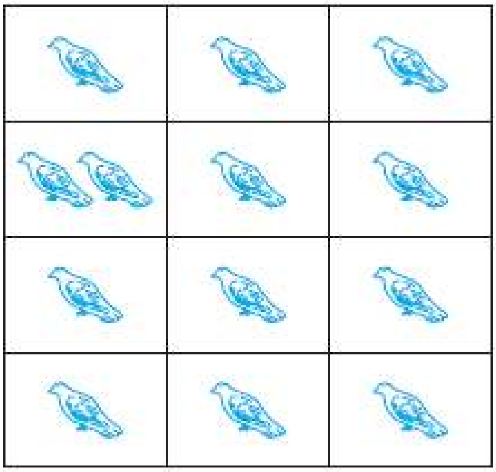
- A one-to-one correspondence is called invertible because we can define an inverse of this function. A function is not invertible if it is not a one-to-one correspondence, because the inverse of such a function does not exist.
- When the composition of a function and its inverse is formed, in either order, an identity function is obtained. To see this, suppose that f is a one-to-one correspondence from the set A to the set B. Then the inverse function f^{-1} exists and is a one-to-one correspondence from B to A. The inverse function reverses the correspondence of the original function, so $f^{-1}(b) = a$ when f (a) = b, and f (a) = b when $f^{-1}(b) = a$.

  Hence, $(f^{-1}Of)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$,

  And $(fOf^{-1})(b) = f(f^{-1}(b)) = f(a) = b$.
- Consequently $(f^{-1}Of) = i_A$ and $(fOf^{-1}) = i_B$, where i_A and i_B are the identity functions on the sets A and B, respectively. That is, $(f^{-1})^{-1} = f$.

❖ **Pigeonhole Principle**

- If k is a positive integer and k+1 objects are placed into k boxes, then at least one of the boxes will contain two or more objects. **OR**
- In mathematics, the pigeonhole principle states that if n items are put into m containers, with n > m, then at least one container must contain more than one item.
- **Proof:** We prove the pigeonhole principle using a proof by contraposition. Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction, because there are at least k+1 objects.



Pigeons in holes. Here there are n = 10 pigeons in m = 9 holes. Since 10 is greater than 9, the pigeonhole principle says that at least one hole has more than one pigeon.

- The abstract formulation of the principle: Let X and Y be finite sets and let f: X → Y be a function.
  - ✓ If X has more elements than Y, then *f* is not one-to-one.
  - ✓ If X and Y have the same number of elements and *f* is onto, then *f* is one-to-one.
  - ✓ If X and Y have the same number of elements and *f* is one-to-one, then *f* is onto.
- If "A" is the average number of pigeons per hole, where A is not an integer then
  - ✓ At least one pigeon hole contains **ceil[A]** (smallest integer greater than or equal to A) pigeons
  - ✓ Remaining pigeon holes contains at most **floor[A]** (largest integer less than or equal to A) pigeons.
- **Example 78:** In a group of 366 people, there must be two people with the same birthday.
- **Example 79:** In a group of 27 English words, at least two words must start with the same letter.
- **Example 80:** How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

  Solution: There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

❖ **Generalized Pigeon hole Principle**

- If n pigeonholes are occupied by Kn+1 or more pigeons then at least one pigeonhole is occupied by K+1 or more pigeons.
- **Example 81:** Find the minimum no of students in a class to be ensure that three of them born in the same month.

Solution: n = 12, K+1 =3 i.e. K=2, Kn+1 = 2*12+1 = 25

- **Example 82:** What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

  Solution: n = 5, K+1 =6 i.e. K=5, Kn+1 = 5*5+1 = 26

- **Example 83:** Show that 7 colours are used to paint 50 bicycles, and then at least 8 bicycles will be of same colour.

  Solution: n = 7, K+1 =8 i.e. K=7, Kn+1 = 7*7+1 = 50

- **Example 84:** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

  Solution: n = 4, K+1 =3 i.e. K=2, Kn+1 = 4*2+1 = 09

- **Example 85:** If (Kn+1) pigeons are kept in n pigeon holes where K is a positive integer, what is the average no. of pigeons per pigeon hole?

  Solution: average number of pigeons per hole = (Kn+1)/n = K + 1/n

  Therefore at least a pigeonholes contains (K+1) pigeons i.e., ceil[K +1/n] and remaining contain at most K i.e., floor[k+1/n] pigeons.

  i.e., the minimum number of pigeons required to ensure that at least one pigeon hole contains (K+1) pigeons is (Kn+1).

- **Example 86:** A bag contains 10 red marbles, 10 white marbles, and 10 blue marbles. What is the minimum no. of marbles you have to choose randomly from the bag to ensure that we get 4 marbles of same color?

  Solution: Apply pigeonhole principle.

  No. of colors (pigeonholes) n = 3 and No. of marbles (pigeons) K+1 = 4

  Therefore the minimum no. of marbles required = Kn+1

  By simplifying we get Kn+1 = 10.

  Verification: ceil[Average] is [Kn+1/n] = 4

  [Kn+1/3] = 4

  Kn+1 = 10 i.e., 3 red + 3 white + 3 blue + 1(red or white or blue) = 10