Unit 1: Theory of Sets

Sets :

A set is collection of well defined objects. An object in set is called a member or ~~set~~ element of set.
Elements of set are usually denoted in lower case
i.e. a, b, c, ...
While sets are denoted by capital letters
A, B, C

- Symbol '∈' indicates the membership in a set.
eg. "If x is an element of A" i.e. $x \in A$

* Ways of describing Sets :
i) Roster form (Listing method)
$A = \{1,2,3,4,5\}$

ii) Statement form
eg. The set of all even integers

iii) Set builder form

* Cardinality : No. of elements present in the sets
eg from above $|A| = 5$

* Symmetric difference
$(A-B) \cup (B-A)$

* Powerset : Set of all subsets of a set is called power set
eg. $A = \{1,2,3\}$
$P_A = \{\{1,2,3\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \phi\}$

No. of elements in a Powerset
$= 2^n$
where n = Cardinality of set

* Null set : Set without any elements
ϕ, { }

* Disjoint sets : When 2 sets don't have a single element in common they are disjoint sets.

Note

* A set is said to be subset of other set if all the elements of that set is present in that set

A = {1, 2} B = {1, 2, 3}
$A \subseteq B$ (A is subset of B)

$B \supseteq A$ (B is superset of A)

* Proper subset : Cardinality of subset is less than that of parent subset.
for a given set there are $2^n - 1$ proper subset

* Infinite set :
eg. {1, 2, 3, 4, ...}
Set of all natural numbers.

* Equal sets
For 2 sets say A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$.
$\therefore A = B$

* Universal sets
A non empty set of which all the sets under

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consideration are subsets is called universal set.
U.

* Singleton set
Set with only 1 element eg. A = { }

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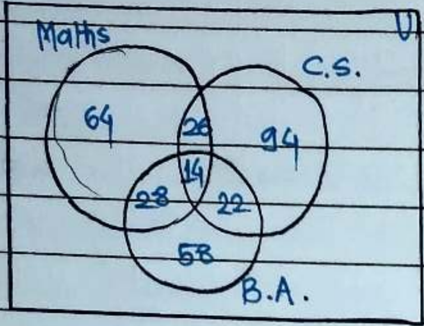
Q. Total No. of students are 260. 64 opted for Maths 94 selected C.S.. 58 selected B.A., 28 students enrolled for Maths & B.A., 26 for Maths & C.S., 22 for C.S. & B.A. and 14 selected all 3. Find students who opted for none.



Here $U = 260$

Now

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

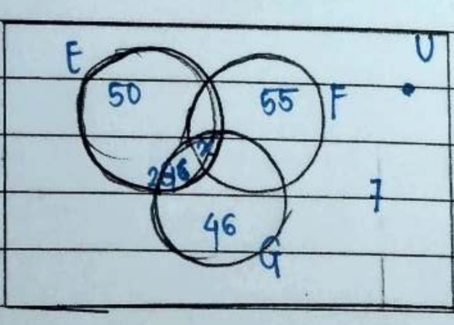
$$\therefore |A \cup B \cup C| = 64 + 94 + 58 - 26 - 22 - 28 + 14$$

$$= 154$$

$\therefore$ $|A \cup B \cup C|$: students who opted for atleast 1 sub. = 154

$\therefore$ Students who opted for none = 106

Q. 80 students are there 50 of them opted for language English, 55 for french, 46 for German, 37 opted for E & F, 28 opted for F & G and 25 E & G, 7 for ~~all~~ none.



i) find students who opted for all

ii) find students who opted for exactly...

→ Here $|A \cup B \cup C| = |E \cup F \cup G| = 80 - 7 = 73$

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Now,

∴ By Principle of inclusion,

$|E \cup F \cup G| = |E| + |F| + |G| - |E \cap F| - |E \cap G| - |F \cap G| + |E \cap F \cap G|$

$\therefore 78 = 50 + 55 + 46 - 37 - 28 - 25 + x$

$\therefore |E \cap F \cap G| = 12$

∴ 12 students opted for all 3

Now for exactly 2

Only 2 = $|E \cap F| + |E \cap G| + |F \cap G| - 3|E \cap F \cap G|$

$= 37 + 28 + 25 - 3 \times 12$

$= 54$

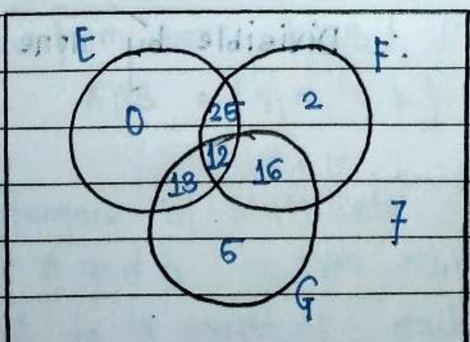
∴ 54 opted for exactly 2

Now for exactly 1

Only 1 = $|~~A \cup B~~\ E \cup F \cup G| - |E \cap F| - |E \cap G| - |F \cap G| + 2|E \cap F \cap G|$

$= 78 - 37 - 28 \mp 25 + 2 \times 12$

∴ Only 1 = 7



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Q Among integers 1 to 1000 How many of them are divisible by 3 ~~or~~ 5 ~~or~~ 7. How many of them are not divisible by 3, 5, 7, How many not divisible by 5 & 7 but by 3.

→

Here let from 1
- divisible by 3 = {A}
- divisible by 5 = B
- divisible by 7 = C

Here $|A| = 333$ $|B| = 200$ $|C| = 142$

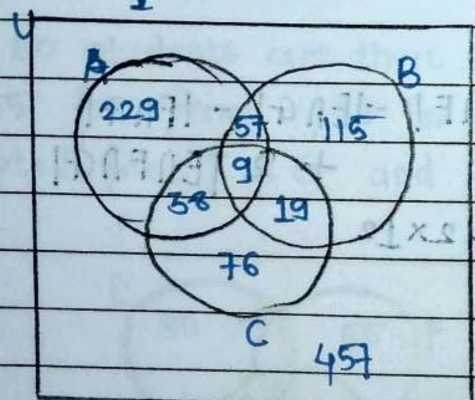
Now

~~|A∪|~~ $|A \cap B|$ = divisible by 3 & 5 $= \frac{1000}{15} = 66$

$|A \cap C|$ = divisible by 3 & 7 $= \frac{1000}{21} = 47$

$|B \cap C|$ = divisible by 5 & 7 $= \frac{1000}{35} = 28$

Also divisible by all 3 $= \frac{1000}{3 \times 5 \times 7} = 9$
$|A \cap B \cap C|$



Now divisible by any of 3
$= |A \cup B \cup C|$
$= 333 + 200 + 142$
$- 47 - 66 - 28 + 9$
$= 543$

∴ Divisible by none = 457

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* Multi set : Collection of elements with one or more occurrence.

- Multiplicity of each element is counted as no. of occurence of that element in a set.
  $A = \{a, a, a, b, c, c, c\}$ $\mu_A(a) = 3$
- Equivalence of multi set : No. of occurence of each element is same in both the set.
- Multisubset
  Multiplicity of each element in a subset must be ≤ parent set
  eg. $\{1, 2, 3, 4\} \subseteq \{1, 1, 2, 3, 3, 4\}$

* Union of multisets
The multiplicity of an element in $A \cup B$ is equal to the maximum of multiplicity of x in A and in B.

eg. $A = \{a, b, c, a, b, c, a, a, a\}$
$B = \{a, a, b, b, b, b\}$

$A \cup B = \{a, a, a, a, a, b, b, b, b, c, c\}$

* Intersection
The multiplicity of an element in $x \in A \cap B$ is equal to the ~~maximum~~ minimum of multiplicity of x in A and in B.

eg. :- from above A & B
$A \cap B = \{a, a, b, b\}$

* Difference of MultiSets
Let A and B be two multisets.
A-B is a multiset such that $x \in A-B$

if $\neg(\mu_A(x) \,\cdot\, \mu_B(x)) \geq 1$

eg1. A = {a, a, b, a}
B = {a, b}

A − B = {a, a}

eg2. A = {1, 2, 3, 4, 2, 2, 3, 3}
B = {1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 4}

A − B = Φ

* Sum of Multisets.

Let A & B be 2 multisets. The sum of A and B is denoted by A+B and defined as: for each

$x \in A+B$, $\mu(x) = \mu_A(x) + \mu_B(x)$

eg. A = {a, b, c, c}   B = {a, a, b, b, c, c}

A + B = {a, a, a, b, b, b, c, c, c}.

Q. A = {a, a, b, c, d, d, d, e}
B = {a, b, d, f, g}   find: A∪B
C = {b, c, e, e, g, h, h}   C∩B
D = {a, d, d, e, f, f, g, h}   B+C
A−D.

ele.	M(x)			
	A	B	C	D
a	2	1	0	1
b	1	1	1	0
c	1	0	1	0
d	3	1	0	2
e	1	0	2	1
f	0	1	0	2
g	0	1	1	1
h	0	0	2	1

$A \cup B = \{a, a, b, c, d, d, e, f, g\}$

$C \cap B = \{b, g\}$

$B + C = \{a, b, b, c, d, e, e, f, g, g, h, h\}$

$A - D = \{a, b, c, d,\}$

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* Propositional Calculus

* **Staments or Propositions.:-**
  - Statement is a declarative sentence which is either true or false but not both
  - The truth or falsity of statement is called its truth value.

* **Laws of formal logic**

There are 2 laws of formal logic.

*1) **Laws of Contradiction**

It states that a statement can't be T/F both

ii) **Law of excluded Middle.**

If p is statement then p is either true / false and there cannot be middle ground

* **Connectives and compound statements**

Sr. No.	Name of connectives	Symbol
1	Negation	~ ¬
2	And (Meet)	$\wedge$
3	Or (Join)	$\vee$
4	if.... then	$\rightarrow$
5	if an only if (iff)	$\leftrightarrow$

* **Compound statement**

Statement formed by primary statements using logical connectives.

eg. p : I am in SE

q : I am learning DM

I am in SE and learning DM

$p \wedge q$

* Negation (simple statement using single)

It is denoted by $\sim p$ or $\neg p$ or $\bar{p}$

Truth table

p	$\sim p$
T	F
F	T

* Conjunction (compound statement)

$p \wedge q$. It is read as "p and q"

Truth table similar to AND gate

* Disjunction

$p \vee q$. It is read as "p or q"

Truth table same as or gate.

* Conditional statement. (if . . . then . . .)

p & q are 2 statements,

denoted as $p \rightarrow q$

read as "if p then q"

Can also be read as

i) p only if q ii) p implies q iii) p if q

iv) p is sufficient for q

- Statement p is hypothesis or antecedent
- Statement q is conclusion or consequent
- if p is true and q is false then $p \rightarrow q$ is false otherwise T.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

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eg.

Let p : n is a natural number

q : n is a rational number

$p : n \in N$ $\quad$ $q : n \in Q$ $\quad$ $p \rightarrow q$: if $n \in N$ then $n \in Q$

T ($n=2$) $\quad$ T ($n \in Q$) $\quad$ T ($\because n=2$ is rational)

T ($n=5$)

* (if and only if) Biconditional statement

denoted as $p \leftrightarrow q$

read as

i) p if and only if q

ii) p iff q

iii) p implies and implied by q

iv) p is necessary and sufficient for q.

- Truth table

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Similar to truth table of X-NOR GATE

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Q: $P(n) \equiv 1^3 + 2^3 + 3^3 \ldots n^3 = \frac{n^2(n+1)^2}{4}$

Step 1. Basis of induction

Put $n=1$

$P(1) = 1 = \frac{1(1+1)^2}{4}$

$= 1$

$\therefore$ P(n) is true for n = 1

Step 2: Induction hypothesis

Put $n=k$

$P(k) = 1^3 + 2^3 + 3^3 \ldots k^3 = \frac{k^2(k+1)^2}{4}$ —— ①

Assume

P(k) is true for n=k

$\therefore$

Step 3: Induction hypothesis

Put $n = k+1$

$P(k+1) = 1^3 + 2^3 + \ldots k^3 + (k+1)^3 = \frac{(k+1)^2 (k+2)^2}{4}$

from —① put value in above eq^n

$\therefore$

$\frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$

$k^2 + 4k + 4 = (k+2)^2$

$\therefore$ LHS = RHS Hence proved

* Special Propositions

$p \to q$ is contiditional statement then

i) $q \to p$ is called its converse

ii) $\sim p \to \sim q$ is called its inverse

iii) $\sim q \to \sim p$ is called its contrapositive

p	q	$p \to q$	$q \to p$	$\sim p$	$\sim q$	$\sim p \to \sim q$
T	T	T	T	F	F	T
F	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

$\sim q \to \sim p$
T
F
T
T

Q. Let p statement denote, "The material is interesting"
q statement denote, "The exercises are challenging"
r statement denote, "The course is enjoyable"

i) Write following in symbolic form

a) The material is interesting and exercises are challenging

b) The material is interesting means the exercises are challenging and conversely.

c) Either materials is interesting or the exercises are not challenging but not the both.

iv) If material is not interesting and exercises are not challenging, then the course is not enjoyable.

i) $p \wedge q$ ii) $(p \to q) \wedge (q \to p)$ iii) $p \oplus \sim q$

iv) $(\sim p \wedge \sim q) \to \sim r$ v) $\sim p \wedge \sim q \wedge \sim r$

Q: $1^2 + 3^2 + 5^2 + \dots (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

→ Basis of induction

Put $n = 1$

$\therefore P(1) : 1 = \frac{1(2-1)(2+1)}{3}$

$\therefore P(1)$ is true for $n = 1$

Step 2 : Induction hypothesis

Put $n = k$

$\therefore P(k) : 1^2 + 3^2 + \dots (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$ —①

Let $P(n)$ is true for $n = k$

Step 3 : Induction step

Put $n = k+1$

$\therefore P(k+1) : 1^2 + 3^2 + \dots (2k-1)^2 + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$

from —① we get

$\frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$

$\therefore k(2k-1) + 6k + 3 = (k+1)(2k+3)$

$\therefore 2k^2 + 5k + 3 = 2k^2 + 5k + 3$

Here LHS = RHS

Hence proved $P(n)$ is true

Q. Express contrapositive, converse & inverse forms of following statement if $3<b$ and $1+1=2$, then $\sin\frac{\pi}{3}=\frac{1}{2}$

→ $p: 3<b$ $q: 1+1=2$ $r: \sin\frac{\pi}{3}=\frac{1}{2}$

Above statement is
$(p \wedge q) \longrightarrow r$

- Contrapositive
$(\sim r \longrightarrow \sim(p \wedge q))$
i.e. if $\sin\frac{\pi}{3} \neq \frac{1}{2}$ then $3 \geq b$ and $1+1 \neq 2$

- Inverse
$\sim(p \wedge q) \longrightarrow \sim r$
i.e. if $3 \geq b$ and $1+1 \neq 2$, then $\sin\frac{\pi}{3} \neq \frac{1}{2}$

- Converse
$r \longrightarrow (p \wedge q)$
i.e. if $\sin\frac{\pi}{3}=\frac{1}{2}$ then $3<b$ and $1+1=2$

* Tautology
A statement formula that is true for all possible values of it's propositional ~~values~~ variables is called a Tautology.
$p \vee \sim p$ is a tautology

* Contradiction
A statement formula that is always false for all possible values of variables is called a contradiction.
$p \wedge \sim p$ is a contradiction.

* Contigency is neither tautology nor Contradiction

- $p \rightarrow p$ is tautology.
- $\sim(p \vee \sim p)$ is contradiction
- $\sim(p \wedge \sim p)$ is ~~co~~ tautology.

Q i) $(p \wedge q) \wedge \sim(p \vee q)$

ii) $(p \leftrightarrow q) \leftrightarrow (q \vee \sim p)$

iii) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

iv) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

→ i) Consider truth table

p	q	p∧q	p∨q	~(p∨q)	(p∧q)∧~(p∨q)
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

~~It~~ is contradiction

iii)

p	q	r	p→q	p→r	q→r	p→(q→r)	(p→q)→(p→r)	(p→(q→r))→((p→q)→(p→r))
T	T	T	T	T	T	T		
T	T	F	T	F	F			
T	F	T	F	T	T			
T	F	F	F	F	T			
F	T	T	T	T	T			
F	T	F	T	T	F			
F	F	T	T	T	T			
F	F	F	T	T	T			

Q. Prove by Mathematical induction
Sum of n natural no.s is $\frac{n(n+1)}{2}$

→ Step 1: Basis of induction
let $n=1$
$\therefore P(1) : 1 = \frac{1(1+1)}{2}$
P(n) is true for $n=1$

Step 2: Induction hypothesis
Let $n=k$
$\therefore P(k) : 1+2+3+\ldots k = \frac{k(k+1)}{2}$
Let P(k) is true for $n=k$

Step 3: Induction step
Let $n=k+1$
$\therefore P(k+1) : 1+2+3 \ldots k+k+1 = \frac{(k+1)(k+2)}{2}$

$\therefore \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$

$\therefore$ LHS = RHS
Hence proved

Q. p : I will study discrete mathematics
q : I will go to movie
r : I am in a good mood
Give ~~logic for~~ statements for following logics
a) $\sim r \rightarrow q$
if I am not in good mood then I will go to movie
b) $\sim q \wedge p$
I will not go to movie and I study DM

c) $q \rightarrow \sim p$

if I will go to movie then I will not study DM

d) $\sim p \rightarrow \sim r$

if I will not study DM then I will not be in good mood.

Q For all integers show that $8^n - 3^n$ is divisible by 5 for $n \geq 1$

→ Here

$P(n) : 8^n - 3^n$

Step 1 : Basis of induction

Now put $n=1$

$\therefore P(1) : 8-3 = 5$

Here $P(1)$ is divisible by 5

$\therefore P(n)$ is true for $n=1$

Step 2 : Induction hypothesis

Now put $n=k$

$\therefore P(k) : 8^k - 3^k$

Assume $8^k - 3^k$ is divisible by 5

$\therefore P(k)$ is true for $n=k$

Step 3 : Induction step

Now put $n = k+1$

$P(k+1) : 8^{k+1} - 3^{k+1}$

$: 8(8^k) - 3(3^k)$

$: (5+3)(8^k) - 3 \cdot 3^k$

$: 5 \cdot 8^k + 3(8^k - 3^k)$

Now here $3(8^k - 3^k)$ is divisible by 5

also $5 \cdot 8^k$ is divisible by 5

$\therefore P(n)$ is true

Q. 100 students are there 45 play cricket, 21 play football, 38 play hockey, 18 play cricket and hockey, 9 play C & F, & 4 play F & H, 23 play none

i) How many play exactly 1 game

ii) How many play exactly 2 games

→ Here 100 students

and 23 play none

∴ students who play atleast 1 = 77

Now by P of I & E

$C \cup F \cup H = C + F + H - C \cap F - F \cap H - C \cap H + C \cap F \cap H$

$\therefore 77 = 45 + 21 + 38 - 18 - 9 - 4 + C \cap F \cap H$

$\therefore C \cap F \cap H = 4$

Now exactly 1 = $C \cup F \cup H - C \cap F - C \cap H - F \cap H + 2\, C \cap F \cap H$

$= 77 - 18 - 9 - 4 + 2 \times 4$

exactly 1 = 54

Now exactly 2 = $C \cap F + C \cap H + F \cap H - 3(C \cap F \cap H)$

$= 18 + 9 + 4 - 3 \times 4$

$= 19$

∴ 19 students play exactly 2

Q. ~~Find~~ Prove $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for $n \geq 1$

1. Basis of Induction

For $n=1$

We have LHS $= 1 + 2^3 + 3^3 = 36$

Here

LHS is divisible by 9

For $n=1$

$P(n)$ is true

2. Induction step

Assume that for $n=k$ $P(n)$ is true

$\therefore k^3 + (k+1)^3 + (k+2)^3$ ~~is~~ $=$ ~~23~~ $9m$

Now we have

$(k+1)^3 + (k+2)^3 + (k+3)^3 =$

$$\left[[k+1]^3 + (k+2)^3 + (k^3 + 27 + 9k^2 + 27k) \right]$$

$$= \left[k^3 + (k+1)^3 + (k+2)^3 + 9(k^2 + 3k + 3) \right]$$

$$= 9(m + (k^2 + 3k + 3))$$

$\therefore$ Assuming

$P(k)$ is true $P(k+1)$ is also true

$\therefore P(n)$ is true for all $n \geq 1$