19/01/24 FRIDAY.

* UNIT : NO: 1 : (ONE) :-

LINEAR D.E. OF HIGHER with constant coefficient.

1.) Basic Formulae of derivative. & integration.

2.) Factorization.

◦ DIFFERENTIATION:-

	$F(x)$	$F'(x) = d/dx$
1.)	x^n	nx^{n-1}
2.)	$\cos x$	$-\sin x$
3.)	$\sin x$	$\cos x$
4.)	$\sec x$	$\sec x \cdot \tan x$.
5.)	$\operatorname{cosec} x$	$\operatorname{cosec} x \cdot \cot x$.
6.)	$\tan x$	$\sec^2 x$
7.)	$\cot x$	$-\operatorname{cosec}^2 x$
8.)	$\log x$	$1/x$.
9.)	$\frac{1}{x}$	$\frac{-1}{x^2}$
10.)	a^x	$a^x \log x$.
11.)	x^x	$x^x \log x$
12.)	K (constant)	0

◦ Basic Differentiation Formulae :-

⟹ ① $\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \frac{du}{dx}$ .... (~~Addition~~) Multiplication Rule

② $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ ......(Division rule)

③ $\frac{d}{dx}\left[f \circ g(x)\right] = \frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

.... (Composite function)

INTEGRATION:-

	$f(x)$	$I = \int f(x) \cdot dx$
1.)	k (constant)	kx
2.)	e^x	e^x
3.)	$\sin x$	$-\cos x$
4.)	$\cos x$	$\sin x$
5.)	$\tan x$	$\log(\sec x)$
6.)	$\cot x$	$\log(\sin x)$
7.)	$\sec x$	$\log(\tan x + \sec x)$
8.)	$\operatorname{cosec} x$	$\log(\operatorname{cosec} x - \cot x)$
9.)	$\cot^2 x$	$\operatorname{cosec}^2 x - \cot x$
10)	$\sec^2 x$	$\tan x$
11.)	$\sec x \cdot \tan x$	$\sec x$.
12.)	$\tan x$	$-\operatorname{cosec}^2 x$
13.)	$1/x$	$\log x$.
14.)	$\log x$	$x \log x - x$.

Integration by Parts:-

(1.) $\int \underline{u} \cdot \underline{v} \, dx = u \int v \, dx - \int \left(\frac{du}{dx} \cdot \int v \, dx \right) dx$

(Differentiate) (Integrate)

⇒ LIATE / ILATE Rule

(2.) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$

(3.) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$

(4.) $\int e^x [f(x) + f'(x)] dx = e^x \cdot f(x)$

(5.) $\int UV\,dx = UV_1 - U'V_2 + U''V_3 - U'''V_4 + U''''V_5 - \ldots$

→ AI AE

e.g.

$$\int \underset{(U)}{x^2}\ \underset{(V)}{\sin 2x}\ dx = x^2\left(\frac{-\cos 2x}{2}\right) - (2x)\left(\frac{-\sin 2x}{4}\right) + (1)\left(\frac{\cos 2x}{8}\right)$$

⇒ $(a-b)^2 = a^2 - 2ab + b^2$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

$a^2 - b^2 = (a+b)(a-b)$

$(a^3 - b^3) = (a-b)(a^2 + ab + b^2) = (a-b)^3 + 3ab(a-b)$

$(a^3 + b^3) = (a+b)(a^2 - ab + b^2) = (a+b)^3 - 3ab(a+b)$

→ $ax^2 + bx + c = 0$ ...(Quadratic Equation)

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \text{ ... roots.}$$

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.

$z = a + ib$, $i = \sqrt{-1}$, $i^2 = -1$

$a = Re(z)$

$b = img(z)$

Differential Equation:- (Defination)

→ An Equation consisting of derivative is called D.E.

Eg: $\frac{dy}{dx} + y = x$

$$\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right) + 3y = e^x$$

Order D.E and Degree of D.E.

LDE with higher order. (Defination)

→ LDE of higher order with constant coefficient is defined as :-

$$\boxed{a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{----} + a_n y = f(x)} \longrightarrow (1)$$

where 'ai' are constants.

When, $F(x) = 0$ then Equation (1) is called Homogenous D.E.

If $f(x) \neq 0$ then Equation (1) is called non-Homogenous L.D.E.

$F(x) = 0$

Simple Form of Equation.

Denote $\frac{d}{dx} = D$

$$\therefore a_0 D^n y + a_1 D^{n-1} y + \text{----} + a_n y = F(x)$$

$$(a_0 D^n + a_1 D^{n-1} + \text{----} + a_n) y = F(x)$$

$$\boxed{\phi(D) y = F(x)}$$

$$\boxed{\phi(D) = a_0 . D^n + a_1 D^{n-1} + \text{----} + a_n}$$

↳ polynomial Differential Operator.

- Auxiliary Equation : $\phi(D) = 0$

Eg:- $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{2x}$

$$(D^2 + 3D + 2) y = e^{2x}$$

AE : $\boxed{D^2 + 3D + 2 = 0}$

SOLUTION OF HOMOGENEOUS D.E.

→ $\phi(D)y = 0$

STEP: 1: Write A.E. $\phi(D) = 0$

STEP: 2: Find roots of an A.E.

STEP: 3: Depending upon the nature of roots, we write solution.

(1) **Roots are Real and Distinct.**

→ $m_1, m_2, m_3, \ldots\ldots, m_n$ are the roots.

Then G.S. is

$$y = c_1 \cdot e^{m_1 x} + c_2 \cdot e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

Eg:- $(D^2 + 3D + 2)y = 0$

AE : $D^2 + 3D + 2 = 0$ ... (Auxiliary Equation)

$D = -1$, $D = -2$ ... (roots)

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x}}$$

TYPE	NATURE OF ROOTS	Complimentary Function.
1.	Real and Distinct	$c_1 e^{m_1 x} + c_2 e^{m_2 x}$
2.	Real and Repeated	$(c_1 x + c_2) \cdot e^{m_1 x}$
3.	Complex	$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$
4.	Complex and Repeated.	$e^{\alpha x}[(c_1 x + c_2)\cos \beta x + (c_1 x + c_2) \sin \beta x]$

(2) If the roots are real and repeated

$m_1 = m_2$

→ $y_1 = (c_1 x + c_2) \cdot e^{m_1 x}$

Eg:- $(D^2 + 4D + 4)y = 0$

A.E. = $D^2 + 4D + 4 = 0$

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$\therefore D = -2, -2 \quad \ldots (\text{roots})$

$\therefore y = (c_1 x + c_2) e^{-2x}$

$$\boxed{y = (c_1 x + c_2) e^{-2x}}$$

If, $\quad m_1 = m_2 = m_3$

$y = (c_1 x^2 + c_2 x + c_3) e^{m_1 x}$

Eg:- $(D^3 + 3D^2 + 3D + 1) y = 0$

A.E. : $D^3 + 3D^2 + 3D + 1 = 0$

$D = -1, -1, -1 \quad \ldots (\text{Roots})$

$$\boxed{y = (c_1 x^2 + c_2 x + c_3) \cdot e^{x}}$$

(3.) If roots are Complex and Distinct.

$\rightarrow \quad \alpha \pm i\beta$

$$\boxed{y = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]}$$

Eg:-

$(D^2 + D + 1) y = 0$

AE : $D^2 + D + 1 = 0$

$D = \dfrac{-1 \pm \sqrt{3} i}{2} \qquad \therefore \alpha = \dfrac{-1}{2}, \quad \beta = \dfrac{\sqrt{3}}{2}$

$$\boxed{y = e^{-x/2} \left[c_1 \cos \left(\frac{\sqrt{3}}{2} \right) x + c_2 \sin \left(\frac{\sqrt{3}}{2} \right) x \right]}$$

(4) If roots are complex and repeated $\alpha \pm i\beta$ repeated 2 times.

$\rightarrow$

$$\boxed{y = e^{\alpha x} [(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x]}$$

Eg:- $(D^2 + 9)^2 y = 0$

AE : $(D^2 + 9)^2 = 0$

$D = \pm 3i, \pm 3i$

$$\boxed{y = (c_1 x + c_2) \cos 3x + (c_3 x + c_4) \sin 3x}$$

SOLUTION OF NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION: (LDE) :-

⇒ The General Solution of a Non-homogeneous LDE is given by,

$$\boxed{y = y_c + y_p}$$

where,

(i.) 'y_c' is the solution of $\phi(D)y = 0$ and it is called a Complimentary Function (C.F.)

(ii.) 'y_p' is a solution of $\phi(D) \cdot y = f(x)$ with no arbitary constant and it is called particular Integral.

o $\boxed{\phi(D) = a_0 \cdot D^n + a_1 \cdot D^{n-1} + \ldots\ldots + a_n}$

.... (Polynomial Differentiation Operator.)

o Inverse Operator $= \dfrac{1}{\phi(D)}$

$$\therefore \phi(D) \left\{ \frac{1}{\phi(D)} \cdot f(x) \right\} = f(x)$$

$$\boxed{y_p = \frac{1}{\phi(D)} \cdot F(x)} \not= \cancel{f(x)}$$

There are 3 methods to generate 'y_p'

(1.) General Method

(2.) Shortcut Method.

(3.) Method of V.P.

GENERAL METHOD :-

→ For $\phi(D) = D$

i.) $\dfrac{1}{D} \cdot f(x) = \displaystyle\int f(x) \cdot dx$

$\dfrac{1}{D^2} \cdot f(x) = \displaystyle\iint F(x)\, dx\, dx$

ii.) $\frac{1}{D-m} \cdot f(x) = e^{mx} \cdot \int e^{-mx} \cdot f(x)\, dx$

iii.) $\frac{1}{D+m} \cdot f(x) = e^{-mx} \cdot \int e^{mx} \cdot f(x)\, dx$

iv.) $\phi(D) = a_0 \cdot D^2 + a_1 \cdot D + a_2 = (D-m_1)(D+m_2)$

$$\frac{1}{a_0 \cdot D^2 + a_1 D + a_2} \cdot f(x) = \frac{1}{(D-m_1)(D+m_2)} \cdot f(x) = \frac{1}{D-m_1}\left[\frac{1}{D-m_2} \cdot f(x)\right]$$

□ SOLVE THE Following using General Method.

1.) $\frac{1}{D-3} \cdot e^{5x}$ Here, $m = +3$

ANS. SOLUTION:- $\frac{1}{D-m} \cdot f(x) = e^{mx} \int e^{-mx} \cdot f(x)\, dx$ ...(Formula)

$$\therefore \frac{1}{D-3} \cdot e^{5x} = e^{+3x} \int e^{-3x} \cdot e^{5x}\, dx$$

$$= e^{3x} \int e^{2x}\, dx = e^{3x}\left(\frac{e^{2x}}{2}\right) = \frac{e^{5x}}{2}$$

$$\boxed{\frac{1}{D-3} \cdot e^{5x} = \frac{e^{5x}}{2}}$$

2.) $\frac{1}{D} \log x$

ANS. SOLUTION:- $\frac{1}{D} \cdot f(x) = \int f(x)\, dx$ ....(FORMULA)

$$\therefore \frac{1}{D} \cdot \log x = \int \log x\, dx = x \log x - x$$

$$\boxed{\frac{1}{D} \log x = x \cdot \log(x) - x}$$

3.) $\frac{1}{D+1} \cdot e^{e^x}$

ANS. SOLUTION:-

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$$\frac{1}{D+m} \cdot F(x) = e^{-mx} \int e^{mx} \cdot F(x)\, dx \quad \ldots \text{(FORMULA)}$$

$$\therefore \frac{1}{D+1} \cdot e^{e^x} = e^{-x} \int e^{x} \cdot e^{e^x}\, dx .$$

Put (Substitute) $\Rightarrow e^x = t$

$e^x dx = dt$.

Let, $I = \int e^t dt = e^t + c = \underline{\underline{e^x}}$

$$\therefore \frac{1}{D+1} \cdot e^{e^x} = e^{-x} \int e^{e^x} dx = e^{-x} \cdot e^{e^x}$$

$$\boxed{\frac{1}{D+1} = e^{-x} \cdot e^{e^x}}$$

4.) $\frac{1}{D+1} \cdot \sin(e^x)$

ANS. SOLUTION:- $\frac{1}{D+m} \cdot F(x) = e^{-mx} \int e^{mx} \cdot F(x) \cdot dx \quad \ldots \text{(FORMULA)}$

$$\therefore \frac{1}{D+1} \cdot \sin(e^x) = e^{-x} \int e^x \cdot \sin(e^x)\, dx$$

Substitute (Put) $e^x = t$

$e^x dx = dt$

$\therefore I = \int \sin(t)\, dt = -\cos(t) + c = \underline{\underline{-\cos(e^x)}}$

$$\frac{1}{D+1} \cdot \sin(e^x) = -e^{-x} \cdot \cos(e^x)$$

$$\boxed{\frac{1}{(D+1)} \cdot \sin(e^x) = -e^{-x} \cdot \cos(e^x)}$$

5.) $(D^2 + 3D + 2)\, y = e^{e^x}$

ANS. SOLUTION:-

The Auxiliary Equation (A.E.) of the given equation is

$D^2 + 3D + 2 = 0$.

The roots of the above equation is:-

$\boxed{D = -1, -2}$

Thus, The General solution of the Differential Equation is given by complimentary Function (C.F.)

$$\therefore \boxed{y = C.F = c_1 . e^{-x} + c_2 . e^{-2x}}$$

$$y_p = \frac{1}{D^2 + 3D + 1} . e^{e^x} = \frac{1}{(D+2)(D+1)} . e^{e^x}$$

$$\therefore y_p = \frac{1}{D^2 + 3D + 1} . e^{e^x}$$

$$= \frac{1}{(D+2)}\left[\frac{1}{(D+1)} . e^{e^x}\right] = \frac{1}{D+2} . e^{-x} . e^{e^x}$$

$$= e^{-2x} \int e^{2x} . e^{-x} . e^{x}\, dx$$

$$\boxed{y_p = e^{-2x} . e^{e^x}}$$

Hence, The complete solution is

$$\boxed{y = y_c + y_p}$$

$$\boxed{y = c_1 . e^{-x} + c_2 . e^{-2x} + e^{-2x} . e^{e^x}}$$

Shortcut Method:-

→ (1.) $f(x) = e^{ax}, k, a^x, a^{-x}$

(2.) $f(x) = \sin(ax+b)$ OR $\cos(ax+b)$

(3.) $f(x) = \sinh ax$ OR $\cosh ax$

(4.) $f(x) = x^m$, m is a +ve integer.

(5.) $f(x) = e^{ax} V$, V is any fn of x.

(6.) $F(x) =$

(7.) $F(x) = xV$, V is any fn of x

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TYPE (1.) $f(x) = e^{ax}$

$\rightarrow$ $\dfrac{1}{\phi(D)} \cdot e^{ax} = \dfrac{1}{\phi(a)} \cdot e^{ax}$, provided $\phi(a) \neq 0$.

Ex:- ① $\dfrac{1}{D^2+4} \cdot e^{-2x}$

Put, $D = -2$

$\therefore = \dfrac{1}{(-2)^2+4} \cdot e^{-2x}$

$P.I. = \dfrac{1}{8} \cdot e^{-2x} = \dfrac{e^{-2x}}{8}$

② $\dfrac{1}{D^2-2D+1} \cdot e^{4x}$

Put, $D = 4$

$= \dfrac{1}{(4)^2-2(4)+1} \cdot e^{4x} = \dfrac{1}{16-8+1} \cdot e^{4x} = \dfrac{1}{9} e^{4x} = \dfrac{e^{4x}}{9}$

③ $\dfrac{1}{D^2+4D+4} \cdot e^{-x}$

$\therefore$ put, $D = -1$

$= \dfrac{1}{(-1)^2+4(-1)+4} e^{-x} = \dfrac{e^{-x}}{1-4+4} = e^{-x}$

④ $\dfrac{1}{D^2-4} \cdot e^{2x}$

Putting, $D=2$ the Denominator becomes zero.

$\therefore$ If $\phi(a) = 0$ then

$\dfrac{1}{\phi(D)} \cdot e^{ax} = x \cdot \dfrac{1}{\phi'(a)} \cdot e^{ax}$, provided $\phi'(a) \neq 0$

Eg:- $\dfrac{1}{D^2-4} \cdot e^{2x} = x \cdot \dfrac{1}{2D-0} \cdot e^{2x} = \dfrac{x \cdot e^{2x}}{2(2)} = \dfrac{x \cdot e^{2x}}{4}$

If $\phi''(a) = 0$ then

$$\frac{1}{\phi(D)} \cdot e^{ax} = x^2 \frac{1}{\phi''(a)} \cdot e^{ax}, \text{ provided } \phi''(a) \neq 0.$$

and so on.

Eg:- $$\frac{1}{D^2-2D+1} e^x = x \frac{1}{2D-2} \cdot e^x = x^2 \frac{1}{2} \cdot e^x = \frac{x^2 \cdot e^x}{2}$$

$\Rightarrow$ $f(x) = k = k \cdot e^{0x}$

$$\frac{1}{\phi(D)} \cdot (k) = \frac{1}{\phi(0)} \cdot k$$

Put, $\boxed{D=0}$

$\Rightarrow$ $f(x) = a^x = e^{\log a^x} = e^{x(\log x)}$

$$\therefore \frac{1}{\phi(D)} \cdot a^x = \frac{1}{\phi(\log a)} \cdot a^x$$

Eg:- $$\frac{1}{D^2+3} \cdot 5^x = \frac{1}{(\log 5)^2+3} 5^x$$

$\Rightarrow$ $f(x) = a^{-x}$

$$\frac{1}{\phi(D)} a^{-x} = \frac{1}{\phi(-\log a)} a^{-x}$$

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Q) SOLVE :- $(D^2+2D+1)\cdot y = e^{-2x} + 4^x + 3$.

ANS. SOLUTION:-

$$(D^2+2D+1)\cdot y = e^{-2x} + 4^x + 3 \text{ ... (Given)} \longrightarrow (1)$$

The A.E. of (1) is $D^2+2D+1=0$

$$\therefore \quad D = -1, -1$$

$$y_c = (c_1 \cdot x + c_2)\cdot e^{-x}$$

Now, $y_p = \dfrac{1}{D^2+2D+1}(e^{-2x} + 4^x + 3)$

$$\therefore y_p = \frac{1}{D^2+2D+1}\cdot e^{-2x} + \frac{1}{D^2+2D+1} 4^x + \frac{1}{D^2+2D+1}\cdot 3$$

$$\therefore y_p = \frac{1}{(-2)^2+(2)(-2)+1} e^{-2x} + \frac{1}{(D+1)^2}\cdot 4^x + \frac{1}{(D+1)^2}\cdot 3$$

$$\therefore y_p = \frac{1}{\cancel{4}-\cancel{4}+1}\cdot e^{-2x} + \frac{1}{(\log 4+1)^2} e^{4x} + \frac{1}{(0+1)^2} \times 3$$

$$\therefore \boxed{y_p = e^{-2x} + \frac{e^{4x}}{(\log 4+1)^2} + 3}$$

$y = y_c + y_p$

Q) $(D^2+4)\cdot y = (e^x+1)^2$

→ SOLUTION:-

$$(D^2+4)y = (e^x+1)^2 \text{ ... (Given)} \longrightarrow (1)$$

The A.E. of (1) is $D^2+4=0$

$$D^2+4=0$$

$$D^2=-4 \qquad D = -2i, +2i \qquad \cancel{\alpha=0} \quad \cancel{\beta=2}$$

$$\boxed{y_c = e^{0x}\left[\underbrace{\cancel{(c_1x+c_2)}}_{c_1}\cdot \cos 2x + \underbrace{\cancel{(c_3x+c_4)}}_{c_2}\cdot \sin 2x\right]}$$

$$y_c = (1)\left[c_1\cdot\cos 2x + c_2 \sin 2x\right]$$

$$\therefore y_p = \frac{1}{(D^2+4)}(e^{2x} + 2e^x + 1)$$

$$\therefore y_p = \frac{1}{(D^2+4)}\cdot e^{2x} + \frac{1}{D^2+4}\cdot 2\cdot e^x + \frac{1}{D^2+4}\cdot(1)$$

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$$y_p = \frac{1}{(2)^2+4}\cdot e^{2x} + 2\cdot\frac{1}{(1)^2+4} e^{x} + \frac{1}{(0)^2+4}(1)$$

$$\therefore y_p = \frac{1\cdot e^{2x}}{8} + \frac{2}{5}e^{x} + \frac{1}{4}$$

$$\boxed{y_p = \frac{e^{2x}}{8} + \frac{2\cdot e^{x}}{5} + \frac{1}{4}} \longrightarrow (2)$$

Hence, The complete solution is:-

$$y_p = y_c + y_p$$

$$\therefore y = c_1\cos 2x + c_2\sin 2x + \frac{e^{2x}}{8} + \frac{2\cdot e^{x}}{5} + \frac{1}{4}$$

TYPE

(2) $f(x) = \sin(ax+b)$ OR $f(x) = \cos(ax+b)$

$\rightarrow$ $D\sin(ax+b) = a\cos(ax+b)$

$D^2\sin(ax+b) = -a^2\sin(ax+b)$

$$\frac{1}{\phi(D^2)}\cdot\sin(ax+b) = \frac{1}{\phi(-a^2)}\cdot\sin(ax+b)$$

$$\therefore \frac{1}{\phi(D^2)}\cdot\cos(ax+b) = \frac{1}{\phi(-a^2)}\cdot\cos(ax+b)$$

NOTE:-

1. Replace D^2 by $-a^2$.
2. Write $D^3 = D^2\cdot D$, $D^4 = (D^2)^2$, so on.
3. Keep D as it is.
4. If Linear term to D is present, in the denominator, then rationalize to get it.

$\Rightarrow$ 1) $\sin^2 x = \frac{1-\cos 2x}{2}$

2) $\cos^2 x = \frac{1+\cos 2x}{2}$

3) $2\sin A\cos B = \cos(A+B) + \cos(A-B)$

4)

5)

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Ex1] $\frac{1}{(D^2+4)} \cdot \sin x.$

$\Rightarrow$ put $D^2 = -a^2 = -(-1)^2 = -1$

$\therefore = \frac{1}{-1+4} \sin x = \frac{1}{3} \sin x$

Ex 2.] $\frac{1}{D^2+2D+1} \cos x$

$\Rightarrow$ put $D^2 = -a^2 = -(-1)^2 = -1$

$\therefore \frac{1}{\cancel{-1}+2D+\cancel{1}} \cos x$

$\therefore = \frac{1}{2} \times \frac{1}{D} \cos x = \frac{1}{2}\int \cos x = \frac{1}{2} \sin x$

$= \frac{1}{2} \sin x$

Ex 3.] $\frac{1}{D^2+4} \cdot \sin 2x$

$\Rightarrow$ put, $\boxed{D^2 = -a^2} = -(2)^2 = -4$

But Denominator becomes zero(0)

$\therefore x \frac{1}{2D+0} \cdot \sin 2x$

$= x \times \frac{1}{2\cancel{(-4)}} \times \frac{1}{D} \cdot \sin 2x = \frac{x}{2}\int \sin 2x = \frac{x}{2}\left(\frac{-\cos 2x}{2}\right)$

$= \frac{-x \cdot \cos 2x}{4}$

Ex4] $\frac{1}{D^3+8} \sin 3x$

$\Rightarrow$ put $D^2 = -a^2 = -(3)^2 = -9$

$\therefore D^3 = D^2 \cdot D = -9D$

$$= \frac{1}{8-9D}\sin 3x = \frac{1}{8-9D} \times \frac{(8+9D)}{(8+9D)}\sin 3x \quad \ldots\text{(Rationalize)}$$

$$= \frac{8+9D}{64-81D^2}\sin 3x$$

We know, $D^2 = -a^2 = -9$

$$\therefore = \frac{8+9D}{64-81(-9)}\sin 3x = \frac{(8+9D)\cdot\sin 3x}{793}$$

$$= \frac{1}{793}(8+9D)\cdot\sin 3x = \frac{1}{793}[8\sin 3x + 9\underline{D\sin 3x}]$$

$$= \frac{1}{793}[8\cdot\sin 3x + 9(3\cdot\cos 3x)]$$

$$= \frac{1}{793}(8\sin 3x + 27\cos 3x)$$

Ex. 5) $\frac{1}{D^2+1}\cdot\sin^2 x$

ANS. SOLUTION:-

$$= \frac{1}{D^2+1}\left[\frac{1-\cos 2x}{2}\right] = \frac{1}{2}\left[\frac{1}{(D^2+1)}\times 1 - \frac{1}{(D^2+1)}\cdot\cos 2x\right]$$

$$\therefore \frac{1}{2}\left[\frac{1}{(0)^2+1}\times 1 - \frac{1}{(-2)^2+1}\cdot\cos 2x\right]$$

$$\therefore \frac{1}{2}\left[1 - \left(\frac{1}{-3}\right)\cos 2x\right]$$

$$= \frac{1}{2}\left[1 + \frac{1\cdot\cos 2x}{3}\right]$$

TYPE

(3.) $F(x) = \sinh ax$ OR $F(x) = \cosh ax$

$\rightarrow$ $\sinh ax = \dfrac{e^{ax} - e^{-ax}}{2}$

$\cosh ax = \dfrac{e^{ax} - e^{-ax}}{2}$

$\dfrac{d}{dx}(\sinh ax) = a \cdot \cosh ax$

$\dfrac{d}{dx}(\cosh ax) = -a \cdot \sinh ax.$

Now, $\dfrac{1}{\phi(D^2)} \cdot \sinh ax = \dfrac{1}{\phi(a^2)} \cdot \sinh ax$ provided $\phi(a^2) \neq 0.$

$\dfrac{1}{\phi(D^2)} \cdot \cosh ax = \dfrac{1}{\phi(a^2)} \cosh ax$ provided $\phi(a^2) \neq 0.$

Simply, Replace D^2 by a^2
(and Follow rules of sine or cosine.)

Ex1.] $\dfrac{1}{D^2+5} \cdot \sinh 2x$

$\Rightarrow$ Put $D^2 = a^2 = (2)^2 = 4$

$= \dfrac{1}{4+5} \sinh 2x$

$= \dfrac{1}{9} \sinh 2x$

Ex 2.] $\dfrac{1}{D^2+2D+1} \cdot \cosh \cancel{2}x$

$\Rightarrow$ Put $D^2 = 1$

$= \dfrac{1}{1+2D+1} \cdot \cosh \cancel{2}x = \dfrac{1}{2+2D} \cosh \cancel{2}x = \dfrac{1-D}{2(1-D^2)} \cosh \cancel{2}x$

$= \dfrac{1}{2}(1-D)\left[\dfrac{1}{1-D^2} \cosh x\right]$

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$$= \frac{1}{2}(1-D)\left[x \frac{1}{-2D} \cdot \cosh x\right]$$

$$= \frac{1}{2}(1-D)\left[\frac{-x}{2} \sinh x\right]$$

$$= \frac{-1}{4}\left[x \cdot \sinh x - D(x \cdot \sinh x)\right]$$

$$= \frac{-1}{4}\left[x \cdot \sinh x - (x \cdot \cosh x + \sinh x)\right]$$

TYPE

(4.) $F(x) = x^m$, m is a +ve integer.

$$\Rightarrow \boxed{\begin{aligned} \frac{1}{1-z} &= 1 + z + z^2 + z^3 + z^4 + \cdots\cdots \\ \frac{1}{1+z} &= 1 - z + z^2 - z^3 + z^4 - \cdots\cdots \end{aligned}}$$

For Eg:- ① $\frac{1}{1+D^2} \cdot x^3$

$$\Rightarrow [1 - D^2 + (D^2)^2] \cdot x^3$$

$$= [1 - D^2 + D^4] \times x^3$$

$$= x^3 - D^2(x^3) + D^4(x^3)$$

$$= x^3 - 6x + 0$$

$$= x^3 - 6x$$

$$\boxed{\frac{1}{1+D^2} \cdot x^3 = x^3 - 6x}$$

② $\frac{1}{D^3+8} \cdot x^4$

ANS. $\frac{1}{8\left[(D^3/8)+1\right]} \cdot x^4$

$$= \frac{1}{8}\left[1 + \frac{D^3}{8}\right]^{-1} \cdot x^4$$

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$$= \frac{1}{8}\left[1 - \frac{D^3}{8} + \left(\frac{D^3}{8}\right)^2\right] x^4$$

$$= \frac{1}{8}\left[x^4 - \frac{1}{8} D^3(x^4)\right]$$

$$= \frac{1}{8}\left[x^4 - \frac{1}{\cancel{8}} \times \cancel{24}^{3}x\right] = \frac{1}{8}\left[x^4 - 3x\right]$$

$$\therefore \boxed{\frac{1}{D^3+8} \cdot x^4 = \frac{1}{8}(x^4 - 3x)}$$

Exercise 1.2

Ex 2) $(D^2 + 13D + 36)y = e^{-4x} + \sinh x.$

ANS. SOLUTION:-

A.E. : $D^2 + 13D + 36 = 0.$

$D = -9, -4$

$$\therefore \boxed{y_c = c_1 \cdot e^{-9x} + c_2 \cdot e^{-4x}} \longrightarrow (1)$$

$$y_p = \frac{1}{D^2+13D+36} \cdot e^{-4x} + \frac{1}{D^2+13D+36} \cdot \sinh x.$$

$$y_p = y_{p_1} + y_{p_2}$$

$$y_{p_1} = \frac{1 \cdot x}{2D + 13 + 0} \cdot e^{-4x}$$

$$\therefore y_{p_1} = \frac{x \cdot e^{-4x}}{-8+13} = \frac{x \cdot e^{-4x}}{5}$$

$$\boxed{y_{p_1} = \frac{x \cdot e^{-4x}}{5}} \longrightarrow (i)$$

$$y_{p_2} = \frac{1}{D^2+13D+36} \cdot \sinh x.$$

Put $D^2 = 1$

$$y_{p_2} = \frac{1}{1+13D+36} \cdot \sinh x = \frac{1}{13D+37} \cdot \sinh x.$$

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$$y_{p_2} = \frac{1}{13D+37} \frac{(13D-37)}{(13D-37)} \cdot \sinh x$$

$$\therefore y_{p_2} = \frac{13D - 37}{169D^2 - 1369} \cdot \sinh x$$

$$\therefore y_{p_2} = \frac{(13D-37)}{169(1)-1369} \cdot \sinh x = \frac{-1}{1200}\left(13D(\sinh x) - 37\sinh x\right)$$

$$\therefore y_{p_2} = \frac{-1}{1200}\left(13 \cdot \cosh x - 37 \cdot \sinh x\right) \longrightarrow (ii)$$

$$\boxed{y_p = \frac{x \cdot e^{-4x}}{5} - \frac{1}{1200}\left(13 \cdot \cosh x - 37\sinh x\right)} \longrightarrow (2)$$

Hence, The complete solution is:-

$$\rightarrow \quad y = c_1 e^{-9x} + c_2 \cdot e^{-4x} + \frac{x \cdot e^{-4x}}{5} - \frac{1}{1200}\left(13 \cdot \cosh x - 37\sinh x\right)$$

Ex 22) $(D^2+1)y = \sin(2x+3) + e^{-x} + 2^x$

ANS. A.E. : $D^3 + 1 = 0$

$$\therefore D = -1, \frac{1 \pm \sqrt{3}}{2}$$

$$y_c = c_1 e^{-x} + e^{1/2 x}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right] \longrightarrow (1)$$

$$y_p = \frac{1}{D^3+1}\sin(2x+3) + \frac{1}{D^2+1} \cdot e^{-x} + \frac{1}{D^3+1} \cdot 2^x$$

$$\therefore y_p = (y_p)_1 + (y_p)_2 + (y_p)_3$$

$$\therefore (y_p)_2 = \frac{1}{D^2+1} \cdot e^{-x} \qquad \text{put } D^2 = -1$$

$$\therefore (y_p)_2 = \frac{1x \cdot e^{-x}}{3D+0} = \frac{1 \cdot e^{-x}}{3D} = \frac{1x}{3}\int e^{-x}\,dx = \frac{1x e^{-x}}{3}$$

$$(y_p)_2 = \frac{x \cdot e^{-x}}{3} \longrightarrow (i)$$

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Exercise 1.2

Ex 19] $(D^3 - 25D)y = \cosh 2x \cdot \sinh 3x$

$\rightarrow$ A.E. : $D^3 - 25D = 0 \Rightarrow D(D^2-25) = 0$

$\therefore \boxed{D = 0, 5, -5}$

Now, C.F. $= y_c = c_1 e^{0x} + c_2 e^{5x} + c_3 e^{-5x}$

$$\boxed{y_c = c_1 + c_2 e^{5x} + c_3 e^{-5x}} \longrightarrow (1)$$

$$P.I. = y_p = \frac{1}{(D^3 - 25D)} \cdot \cosh 2x \cdot \sinh 3x$$

$$\therefore y_p = \frac{1}{D^3 - 25D}\left[\frac{\sinh 5x + \sinh x}{2}\right]$$

$$\therefore y_p = \frac{1}{2}\left[\frac{1}{D^3 - 25D} \cdot \sinh 5x + \frac{1}{D^3 - 25D} \cdot \sinh x\right]$$

$$\therefore y_p = \frac{1}{2}\left[x \cdot \frac{1}{3D^2 - 25} \cdot \sinh 5x + \frac{1}{1 - 25(1)} \cdot \sinh x\right]$$

$$\therefore y_p = \frac{1}{2}\left[\frac{x}{3(25) - 25} \cdot \sinh 5x - \frac{1}{24} \cdot \sinh x\right] = \frac{1}{2}\left[\frac{x}{50} \cdot \sinh 5x - \frac{1}{24} \sinh x\right]$$

$$\therefore y_p = \frac{1}{2}\left[\frac{x}{50} \cdot \sinh 5x - \frac{1}{24} \cdot \sinh x\right]$$

$$\boxed{y_p = \frac{1}{2}\left[\frac{x}{50} \cdot \sinh 5x - \frac{1}{24} \cdot \sinh x\right]} \longrightarrow (2)$$

Hence, The Complete solution is:-

$$y = y_c + y_p$$

$$\therefore y = c_1 + c_2 \cdot e^{5x} + c_3 \cdot e^{-5x} + \frac{1}{2}\left[\frac{x}{50} \cdot \sinh 5x - \frac{1}{24} \sinh x\right]$$

$$\boxed{y = c_1 + c_2 \cdot e^{5x} + c_3 \cdot e^{-5x} + \frac{1}{2}\left[\frac{x}{50} \cdot \sinh 5x - \frac{1}{24} \cdot \sinh x\right]}$$

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TYPE

(5) $f(x) = e^{ax}.V$, V is any function of x.

$\rightarrow$ $D(e^{ax}.V) = D(e^{ax})V + e^{ax}DV$

$= a.e^{ax}.V + e^{ax}.DV$

$D(e^{ax}V) = e^{ax}(D+a).V$

$D^2(e^{ax}.V) = e^{ax}.(D+a)^2.V$

$$\boxed{\frac{1}{\phi(D)}e^{ax}V = e^{ax}\frac{1}{\phi(D+a)}.V}$$

Ex. 1] $(D^2 + 2D + 1)y = e^{-x}.\sin 2x$

ANS. SOLUTION:-

$$y_p = \frac{1}{(D^2+2D+1)}.e^{-x}.\sin 2x$$

$$\therefore y_p = e^{-x}\frac{1}{(D-1)^2+2(D-1)+1}.\sin 2x$$

$$\therefore y_p = e^{-x}\left(\frac{1}{D^2}.\sin 2x\right) \quad \text{put } D^2 = -4$$

$$\therefore \boxed{y_p = e^{-x}\left(\frac{-1}{4}\sin 2x\right)} \longrightarrow (2)$$

$y_c \Rightarrow$ A.E. : $D^2 + 2D + 1 = 0$.

$$\boxed{D = -1, +1}$$

$$\therefore \boxed{y_c = c_1.e^{-x} + c_2.e^{x}} \longrightarrow (1)$$

Hence, $y = y_c + y_p$

$$\therefore \boxed{y = c_1.e^{-x} + c_2.e^{x} + \frac{-e^{-x}.\sin 2x}{4}}$$

Ex 2.] $\frac{1}{\phi(D)}.e^{ax}.V = e^{ax}\frac{1}{(D+a)}.V$ ... (Formula)

$\rightarrow$ $(D^2 + 6D + 9)y = \frac{e^{-3x}}{x^2}$.

ANS. SOLUTION:- A.E. : $D^2 + 6D + 9 = 0$.

$$\therefore \boxed{D = -3, -3}$$

$\therefore y_c = (c_1.x + c_2).e^{-3x} \longrightarrow (1)$

$$y_p = \frac{1}{D^2+6D+9}\cdot\left(\frac{e^{-3x}}{x^2}\right)$$

$$\therefore\ y_p = e^{-3x}\frac{1}{(D-3)^2+6(D-3)+9}\cdot x^{-2}$$

$$\therefore\ y_p = e^{-3x}\frac{1}{D^2}\cdot x^{-2}$$

$$\therefore\ y_p = e^{-3x}\iint x^{-2}\,dx\,dx = e^{-3x}\int\frac{x^{-1}}{-1}\,dx = -e^{-3x}\log x$$

$$\boxed{y_p = -e^{-3x}\log x}$$

Hence, $y = (c_1x+c_2)\cdot e^{-3x} - e^{-3x}\cdot\log x$.

$$\boxed{y = e^{-3x}\left[(c_1x+c_2) - \log x\right]}$$

TYPE

(6.) $f(x) = x\cdot V$, V is any function of x.

→ $$\boxed{\frac{1}{\phi(D)}\cdot x\cdot V = \left[x - \frac{\phi'(D)}{\phi(D)}\right]\frac{1}{\phi(D)}\cdot V}$$

This Rule is called 'xV' rule.

This is applicable only if

1.) The power of x is one.

2.) $\frac{1}{\phi(D)}\cdot V$ is not a case of Failure.

Eg.) $\frac{1}{D^2+1}\sin x \xrightarrow{\text{put } D^2=-1}$ Den $=0$ $\Longrightarrow$ Case of Failure.

Eg:- $\frac{1}{D^2+4}\cdot x\cdot\sin 3x$

ANS:- $\left[x - \frac{2D}{D^2+4}\right]\frac{1}{D^2+4}\cdot\sin 3x$

put $D^2=-9$

$$= \left[x - \frac{2D}{-9+4}\right]\frac{1}{-9+4}\cdot\sin 3x$$

$$= \frac{-1}{5}\left[x+\frac{2D}{5}\right]\sin 3x = \frac{-1}{5}\left[x\sin 3x + \frac{2\times 3\cos 3x}{5}\right]$$

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METHOD OF VARIATION OF PARAMETER:-

→ Consider,

$$a_0 \cdot \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 \cdot y = F(x)$$

A.E. : $a_0 \cdot D^2 + a_1 D + a_2 = 0$

$$\boxed{y_c = c_1 y_1 + c_2 y_2}$$

By using method of variation of parameter

$$\boxed{y_p = u \cdot y_1 + v y_2}$$

$$\boxed{W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$\boxed{u = \int \frac{-y_2 \cdot F(x)}{W} dx} \qquad \boxed{v = \int \frac{+y_1 \cdot F(x)}{W} dx}$$

Ex 1) $(D^2+1)y = \sec x$

ANS. A.E. : $D^2+1 = 0$

$\therefore D = \pm i$

$\therefore \boxed{y_c = c_1 \cdot \cos x + c_2 \cdot \sin x.} \longrightarrow (1)$

Here, $y_1 = \cos x \qquad y_2 = \sin x$

$\therefore W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$\boxed{W = 1}$

$u = \int \frac{-y_2 \cdot F(x)}{W} dx$

$\therefore u = -\int \frac{\sin x \cdot \sec x}{1} dx = -\int \tan x \, dx = -\log(\sec x)$

$v = \int \frac{+y_1 \cdot F(x)}{W} dx$

$\therefore v = +\int \frac{\cos x \cdot \sec x}{1} dx = +\int 1 \, dx = x$

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$\therefore y_p = u \cdot y_1 + v y_2$

$\therefore y_p = -\log|\sec x| \cdot \cos x + x \cdot \sin x$

$$\boxed{y_p = -\log(\sec x) \cdot \cos x + x \cdot \sin x} \longrightarrow (2)$$

Hence, complete solution is:-

$y = y_c + y_p$

$\therefore y = c_1 \cos x + c_2 \cdot \sin x - \log(\sec x) \cdot \cos x + x \sin x.$

Ex 2. $(D^2+1) \cdot y = x \cdot \sin x.$

ANS. A.E. $D^2+1 = 0.$

$\therefore D = \pm i$

$$C.F. = \boxed{y_c = c_1 \cdot \cos x + c_2 \cdot \sin x} \longrightarrow (1)$$

Here, $y_1 = \cos x \qquad y_2 = \sin x.$

$y_1' = -\sin x \qquad y_2' = \cos x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x) = \cos^2 x + \sin^2 x = 1$$

$\boxed{W = 1}$

$$u = -\int \frac{\sin x \cdot x \sin x}{1} dx = -\int x \cdot \sin^2 x \, dx = -\int x\left(\frac{1-\cos 2x}{2}\right) dx$$

$$\therefore u = \frac{1}{2}\left[\int x\,dx - \int x \cdot \cos 2x \, dx\right] = \frac{1}{2}\left[\frac{x^2}{2} - \left(\frac{x \cdot \sin 2x}{2} + \frac{\cos 2x}{4}\right)\right]$$

$$v = \int \frac{y_1 \cdot f(x)}{W} dx = \int \frac{\cos x \cdot x \sin x \, dx}{1} = \frac{1}{2}\int x \cdot \sin 2x \, dx$$

$$\therefore v = \frac{1}{2}\left[x\left(\frac{-\cos 2x}{2}\right) + \frac{\sin 2x}{4}\right] = \frac{1}{2}\left[\frac{\sin 2x}{4} - \frac{x \cos 2x}{2}\right]$$

$$\therefore y_p = \frac{\cos x}{2}\left[\frac{x^2}{2} - \left(\frac{x \cdot \sin 2x}{2} + \frac{\cos 2x}{4}\right)\right] + \frac{\sin x}{2}\left[\frac{\sin 2x}{4} - \frac{x \cdot \cos 2x}{2}\right] \longrightarrow (2)$$

Hence, The complete solution is:-

$$\boxed{y = c_1 \cos x + c_2 \sin x + \frac{\cos x}{2}\left[\frac{x^2}{2} - \left(\frac{x \cdot \sin 2x}{2} + \frac{\cos 2x}{4}\right)\right] + \frac{\sin x}{2}\left[\frac{\sin 2x}{4} - \frac{x \cdot \cos 2x}{2}\right]}$$

Ex 3: $(D^2+3D+2)y = \sin(e^x)$

ANS. A.E. : $D^2+3D+2=0.$

$\therefore \boxed{D = -1, -2}$

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

Here, $y_1 = e^{-x}$ $\qquad y_2 = e^{-2x}$

$y_1' = e^{-x}$ $\qquad y_2' = -2e^{-2x}$

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$$

$$\boxed{W = -e^{-3x}}$$

$$u = \int \frac{-y_2 \cdot f(x)}{W} dx$$

$$\therefore u = \int \frac{-e^{-2x} \cdot \sin(e^x)}{-e^{-3x}} dx = \int e^x \cdot \sin(e^x)\, dx$$

Put $e^x = t \Rightarrow e^x dx = dt$

$$\therefore u = \int \sin t\, dt = -\cos t$$

$$\boxed{u = -\cos(e^x)}$$

$$v = \int \frac{+y_1 \cdot F(x)}{W} dx$$

$$\therefore v = \int \frac{e^{-x} \cdot \sin(e^x)}{-e^{-3x}} dx$$

Put, $e^x = t \Rightarrow e^x dx = dt.$

$$\therefore v = -\int t \cdot \sin t\, dt$$

$$\therefore v = -[t(-\cos t) + \sin t]$$

$$\therefore \boxed{v = e^x \cos(e^x) - \sin(e^x)}$$

$$\therefore y_p = uy_1 + vy_2$$

$$\therefore \boxed{y_p = -\cos(e^{-x}) \cdot e^{-x} + [e^x \cdot \cos(e^x) - \sin(e^x)] \cdot e^{-2x}}$$

Hence, The complete solution is:-

$y_G = y_c + y_p$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + [e^x \cdot \cos(e^x) - \sin(e^x)] \cdot e^{-2x} - \cos(e^x)\, e^{-x}}$$

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Ex 5] $(D^2+4)y = 4\cdot\sec^2 2x$.

ANS. A.E. : $D^2+4=0$

$D = \pm 2i$

$$\boxed{y_c = c_1\cdot\cos 2x + c_2\cdot\sin 2x}$$

$y_1 = \cos 2x$, $y_2 = \sin 2x$

$y_1' = -2\sin 2x$ $y_2' = 2\cos 2x$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$u = \int \frac{-y_2\cdot F(x)}{W}\cdot dx$$

$$\therefore u = -\int \frac{\sin 2x\cdot 4\sec^2 2x}{2}\, dx$$

$$\therefore u = -2\int \sec 2x\cdot\tan 2x\, dx = \frac{-2\cdot\sec 2x}{2} = -\sec 2x$$

$$\boxed{u = -\sec 2x} \longrightarrow (1)$$

$$v = \int \frac{y_1\cdot F(x)}{W}\, dx$$

$$\therefore v = \int \frac{\cos 2x\cdot 4\cdot\sec^2 2x}{2}\, dx$$

$$\therefore v = 2\int \sec 2x\, dx$$

$$\boxed{v = \log(\sec 2x + \tan 2x)} \longrightarrow (2)$$

$$\boxed{y_p = \cos 2x(-\sec 2x) + \sin 2x\,[\log(\sec 2x + \tan 2x)]}$$

Hence, complete solution

$y = y_c + y_p$

$y = c_1\cdot\cos 2x + c_2\cdot\sin 2x + \cos 2x(-\sec 2x) + \sin 2x\cdot[\log(\sec 2x + \tan 2x)]$

Ex 4.) $\dfrac{d^2y}{dx^2} - 2\cdot\dfrac{dy}{dx} = e^x\cdot \sin x$

ANS. SOLUTION:- $D = \dfrac{d}{dx}$

$\therefore \; D^2y - 2Dy = e^x\cdot\sin x.$

$(D^2 - 2D)\,y = e^x.\sin x$

A.E. : $D^2 - 2D = 0.$

$\therefore \boxed{D = 0, 2}$

C.F. $= y_c = A\cdot e^{0x} + B\cdot e^{2x}$

$\boxed{y_c = A + B\cdot e^{2x}} \longrightarrow (1)$

$\therefore y_p = \dfrac{1}{D^2 - 2D}\cdot e^x\cdot\sin x$

~~∴ yp = 1~~

$y_1 = 1 \qquad y_2 = e^{2x}$

$y_1' = 0 \qquad y_2' = 2e^{2x}$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x} - 0 = \underline{\underline{2\cdot e^{2x}}}$

$\boxed{W = 2e^{2x}}$.

$u = \displaystyle\int \frac{-y_2\cdot F(x)\,dx}{W}$

$\therefore u = \displaystyle\int \frac{-e^{2x}\cdot e^x\cdot\sin x\,dx}{2e^{2x}} = -\int \frac{\cancel{e^{2x}}\cdot e^x\sin x\,dx}{2\cdot \cancel{e^{2x}}} = \frac{-1}{2}\int e^x\sin x\,dx$

$\therefore u = \dfrac{-1}{2}\left[\dfrac{e^x(\sin x - \cos x)}{1+1}\right] = \dfrac{-1}{4}\cdot e^x(\sin x - \cos x)$

$\boxed{u = -\dfrac{1}{4}\cdot e^x(\sin x - \cos x)} \longrightarrow (i)$

$v = \displaystyle\int \frac{y_1\cdot F(x)\,dx}{W}$

$\therefore v = \displaystyle\int \frac{1\cdot e^x\cdot\sin x\,dx}{2\cdot e^{2x}} = \frac{1}{2}\int e^{-x}\cdot\sin x\,dx$

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$$v = \frac{1}{2}\left[\frac{e^{-x}}{1+1}(-\sin x - \cos x)\right]$$

$$\therefore \boxed{v = \frac{-1}{4}\cdot e^{-x}(\sin x + \cos x)} \longrightarrow \text{(ii)}$$

$$\therefore y_p = y_1 \cdot u + y_2 \cdot v.$$

$$\therefore y_p = \frac{-1}{4}\cdot e^{x}(\sin x - \cos x) - \frac{e^{2x}}{4}\cdot e^{-x}\cdot(\sin x + \cos x)$$

$$\therefore y_p = \frac{-1}{4} e^{x}\cdot\sin x + \cancel{\frac{1}{4}\cdot e^{x}\cos x} - \frac{e^{x}\sin x}{4} - \cancel{\frac{e^{x}\cos x}{4}}$$

$$\therefore y_p = \frac{-2\cdot e^{x}\cdot\sin x}{4} = \frac{-1}{2}\cdot e^{x}\cdot\sin x$$

$$\boxed{y_p = \frac{-1}{2}\cdot e^{x}\cdot\sin x} \longrightarrow (2)$$

Hence, The Complete Solution is:-

$$y = y_c + y_p$$

$$\therefore y = A + B\cdot e^{2x} - \frac{1}{2}\cdot e^{x}\cdot\sin x$$

$$\Rightarrow \boxed{y = A + B\cdot e^{2x} - \frac{1}{2}\cdot e^{x}\cdot\sin x}$$

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Cauchy's LDE :-

→ Defination: The Equation is of the form

$$a_0 x^n \frac{d^x n}{dy^x} + a_1 \cdot x^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}y} + \ldots + a_n y = f(x)$$

is called a cauchy's LDE. ⟶ ①

For Eg:- $2x^2 \frac{d^2y}{dx^2} + 3x \cdot \frac{dy}{dx} + y = \sin[\log x]$

Convert Equation ① into LDE with constant coefficient by using substitution $\boxed{x = e^z}$

⟶ $\boxed{z = \log x}$ $\therefore \frac{dz}{dx} = \frac{1}{x}$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = \frac{dy}{dz} \times \frac{1}{x}$$

$$\therefore x \cdot \frac{dy}{dx} = \frac{dy}{dz}, \quad D = \frac{d}{dz}$$

similarly, $\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$

$$\therefore x^2 \frac{d^2y}{dx^2} = (D^2 - D)y = D(D-1)y$$

$$\therefore x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$ and so on.

□ Exercise 1.4

1.) $x^2 \cdot \frac{d^2y}{dx^2} - 4x \cdot \frac{dy}{dx} + 6y = x^5$

ANS. SOLUTION:- Put $x = e^z$

$$\therefore \boxed{z = \log x}$$

$$\therefore x \cdot \frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

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$\therefore\ D(D-1)y - 4Dy + 6y = e^{5z}$

$\therefore\ (D^2 - D - 4D + 6)y = e^{5z}$

$\therefore\ (D^2 - 5D + 6)y = e^{5z}$

A.E. : $D^2 - 5D + 6 = 0$

$\therefore\ D = 2, 3$

$\therefore\ y_c = c_1 \cdot e^{2x} + c_2 \cdot e^{3x}$

$y_p = \dfrac{1}{D^2 - 5D + 6} \cdot e^{5z}$

put, $D = 5$

$\therefore\ y_p = \dfrac{1}{25 - 25 + 6} \cdot e^{5z} = \dfrac{1 \cdot e^{5z}}{6}$

$$\boxed{y_p = \frac{e^{5z}}{6}}$$

$\therefore\ y = c_1 \cdot e^{2x} + c_2 \cdot e^{3x} + \dfrac{e^{5z}}{6}$

$$\therefore\ \boxed{y = c_1 \cdot x^2 + c_3 \cdot x^3 + \frac{x^5}{6}}$$

(2.) $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = x^2 \cdot \log x.$

ANS. SOLUTION:- Put, $x = e^z$

$\therefore\ z = \log x$

$\therefore\ x\dfrac{dy}{dx} = Dy$

$\therefore\ x^2 \dfrac{d^2y}{dx^2} = D(D-1)y.$

$\therefore\ D(D-1)y - 3Dy + 5y = e^{2z} \cdot y.$

$\therefore\ (D^2 - 4D + 5)y = z \cdot e^{5z}$

A.E. : $D^2 - 4D + 5 = 0$

$\therefore\ D = 2 \pm i \qquad \alpha = 2 \quad \beta = 1$

$\therefore\ y_c = e^{2x}\left[c_1 \cos z + c_2 \cdot \sin z\right]$

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$$\therefore y_p = \frac{1}{D^2-4D+5} \cdot z \cdot e^{2z}$$

$$\therefore y_p = e^{2z}\left(\frac{1}{(D+2)^2-4(D+2)+5} z\right)$$

$$\therefore y_p = e^{2z} \cdot \left(\frac{1}{D^2 \cancel{+4D}+4 \cancel{-4D}-8+5}\right) \cdot z$$

$$\therefore y_p = e^{2z}\left(\frac{1}{1+D^2} \cdot z\right) = e^{2z}(1-D^2)\cdot z = e^{2z} \cdot z$$

$$\boxed{y_p = e^{2z} \cdot z}$$

$$\therefore y = e^{2x}[c_1 \cdot \cos z + c_2 \cdot \sin z] + e^{2z} \cdot z.$$

$$\therefore y = e^{2x}[c_1 \cdot \cos(\log x) + c_2 \cdot \sin(\log x)] + x^2 \cdot (\log x)$$

$$\rightarrow \boxed{y = e^{2x}[c_1 \cdot \cos(\log x) + c_2 \cdot \sin(\log x)] + x^2(\log x)}$$

(3.) $$x^3 \cdot \frac{d^3y}{dx^3} + 3x^2 \cdot \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y \cancel{-6} = 65\cos[\log x]$$

ANS. SOLUTION:- Put, $x = e^z$

$$\therefore z = \log x$$

$$\therefore x\frac{dy}{dx} = Dy.$$

$$\therefore x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$\therefore x^3 \cdot \frac{d^3y}{dx^3} = D(D-1)(D-2)y.$$

$$\rightarrow D(D-1)(D-2)y + 3D(D-1)y + Dy + 8y = 65\cos z$$

$$\therefore (D^3+3)y = 65\cos z$$

A.E. : $D^3 + 3 = 0$

$$\therefore D = -2,\ 1 \pm i\sqrt{3}$$

$$\therefore y_c = c_1 \cdot e^{-2z} + e^z[c_2 \cdot \cos\sqrt{3}z + c_3 \cdot \sin\sqrt{3}z]$$

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$\therefore y_p = \frac{1}{D^3+8} \cdot 65\cos z$

$\therefore y_p = \cancel{\cos}\ 65\left[\frac{1}{D^3+8} \cdot \cos z\right]$

put, $D^2 = -1$

$\therefore D^3 = D^2 \cdot D = -D$

$\therefore y_p = \frac{65}{8-D} \cdot \cos z.$

$\therefore y_p = \frac{65}{8-D} \times \frac{(8+D)}{(8+D)} \cdot \cos z = 65\left[\frac{8+D}{64-D^2} \cdot \cos z\right]$

$\therefore y_p = (8+D)\cdot \cos z$

$\therefore y_p = 8\cos z + D(\cos z) = 8\cos z + (-\sin z)$

$y_p = 8\cos z - \sin z$

~~y_p~~

$y = c_1 \cdot e^{-2z} + e^{z}[c_2 \cdot \cos\sqrt{3}z + c_3 \cdot \sin\sqrt{3}z] + 8\cos z - \sin z$

$\rightarrow \therefore \boxed{y = c_1 \cdot x^{-2} + x[c_2 \cdot \cos(\sqrt{3}\log x) + c_3 \cdot \sin(\sqrt{3}\log x)] + 8\log x - \sin\log x}$

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LEGENDRE'S LDE :-

→ Defination:-

$$a_0(ax+b)^n \cdot \frac{d^ny}{dx^n} + a_1(\cancel{\cos}\,ax+b)^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_ny = F(x)$$

Put , $ax+b = e^z$

$\therefore z = \log(ax+b)$

$$(ax+b)\frac{dy}{dx} = a \cdot Dy$$

$$(ax+b)^2 \cdot \frac{d^2y}{dx^2} = a^2 \cdot D(D-1)\,y.$$

$$(ax+b)^3 \cdot \frac{d^3y}{dx^3} = a^3 \cdot D(D-1)(D-2)\,y.$$

Exercise 1.4

Q] 29 $(x+2)^2 \cdot \frac{d^2y}{dx^2} + 3(x+2)\frac{dy}{dx} + y = 4\sin[\log(x+2)]$

ANS. SOLUTION:- Put $x+2 = e^z$

$\therefore z = \log(x+2)$

$\therefore (x+2)\frac{dy}{dx} = Dy \;(1)$

$\therefore (x+2)^2 \cdot \frac{d^2y}{dx^2} = D(D-1)\,y \;(1)^2$

$\therefore (D(D-1) + 3D + 1)\,y = 4\sin z$

$\therefore (D^2 + 2D + 1)\,y = 4 \cdot \sin z$

A.E. : $D^2 + 2D + 1 = 0.$

$\therefore D = -1, -1.$

$\therefore y_c = (c_1 \cdot z + c_2) \cdot e^{-z} \longrightarrow$ (i)

$\therefore y_p = \frac{1}{D^2+2D+1} \cdot 4\sin z$

Put , $D^2 = -1$

$\therefore y_p = \frac{-1}{\cancel{2}D} \cdot \cancel{4}^{2}\sin z = 2\int \sin z\,dz = -2\cos z$

$y_p = -2\cos z \longrightarrow$ (ii)

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$$\therefore y = (c_1 \cdot z + c_2) \cdot e^{-z} + (-2\cos z)$$

$$\longrightarrow \boxed{y = (c_1 \cdot \cancel{z}\log x + c_2) \cdot x^{-2} - 2\cos(\log x)}$$

Q. 33: $(x+1)^2 \cdot \dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} - y = 2\log(x+1) + x - 1$

ANS. **SOLUTION:-** put, $x+1 = e^z$

$$\therefore z = \log(x+1)$$

$$\therefore (x+1) \cdot \frac{dy}{dx} = (1)\, Dy$$

$$\therefore (x+1)^2 \cdot \frac{d^2y}{dx^2} = (1)^2 D(D-1)y$$

$$\therefore (D(D-1)y + Dy - y) = 2 \cdot \log(x+1) + x - 1$$

$$\therefore (D^2 - D + D - 1)y = 2\log(x+1) + x - 1$$

$$\therefore (D^2 - 2D - 1)\, y = 2z + e^z - 1 - 1 = 2z + e^z - 2$$

$$\boxed{(D^2 - 2D - 1)\, y = 2z + e^z - 2} \longrightarrow (1)$$

Now, A.E : $D^2 - 2D - 1 = 0$

$$\therefore D = +1, -1.$$

$$\therefore y_c = c_1 \cdot e^x + c_2 \cdot e^{-x} \longrightarrow (i)$$

$$\therefore y_p = \frac{1}{(D^2-1)}(2z + e^z - 2)$$

$$\therefore y_p = \frac{1}{(D^2-1)} \cdot 2z + \frac{1}{(D^2-1)} \cdot e^z - \frac{1}{(D^2-1)}(2)$$

$$\therefore y_p = 2 \cdot \frac{1}{(D^2-1)} \cdot z + \cancel{2} \cdot \frac{1}{2D} \cdot e^z - \frac{1}{(0-1)} 2$$

$$\therefore y_p = +2 \frac{1}{(D^2 - D^2_1)} \cdot z + \cancel{2} \frac{1}{2D} \cdot e^z + 2$$

$$\therefore y_p = 2 \frac{1}{(-1+D^2)} z + \frac{e^z}{2D} + 2 = \frac{-2z}{(1-D^2)} + \frac{1}{2}\int e^z dz + \ldots$$

$$\therefore y_p = -2z(1 + \cancel{D^2 + (D^2)^2}) + \frac{e^z}{2} + 2 = -2z + \frac{e^z}{2} + 2$$

$$y_p = -2\log(x+1) + \frac{(x+1)}{2} + 2 \longrightarrow (ii)$$

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Hence, The complete solution of the LDE is:-

$$\boxed{y = c_1 e^{x} + c_2 e^{-x} - 2\log(x+1) + \frac{(x+1)}{2} + 2.}$$