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Fourier Transform

Fourier Series :- $f(x)$ is periodic f^n of period $2L$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

$x \in (0, 2L)$ OR $(-L, L)$

$$a_0 = \frac{1}{2L} \int_0^{2L} f(x)\, dx \quad , \quad a_n = \frac{1}{2L} \int_0^{2L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{2L} \int_0^{2L} f(x) \sin \frac{n\pi x}{L}\, dx$$

$$\boxed{f(x)} \xrightarrow[\text{Transform}]{\text{Fourier}} \boxed{F(\lambda)}$$

eg:- time domain $\xrightarrow{\text{Trans.}}$ Frequency domain

Fourier transform $\longrightarrow$ 1. Solve pde
↳ $f(x)$ continuous f^n 2. Solve integral equations.

Z-transform $\longrightarrow$ Solve difference eq^n
↳ $\{f(k)\}$ discrete valued f^n.

Def^n:- Let $f(x)$ be a f^n defined $(-\infty, \infty)$, then $F\{f(x)\}$ is defined as

$$\boxed{F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx = F(\lambda) \quad \text{—— (1)}}$$

Ex:- Find F.T of $f(x)$, $f(x) = e^{-5x}$, $x > 0$

Sol^n:- $F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx$

$$= \int_0^\infty e^{-5x} e^{-i\lambda x} \, dx$$

$$= \int_0^\infty e^{-(5+i\lambda)x} \, dx = \left[\frac{e^{-(5+i\lambda)x}}{-(5+i\lambda)} \right]_0^\infty$$

$$F(\lambda) = \frac{-e^{-\infty} + e^{0}}{5+i\lambda} = \frac{1}{5+i\lambda}$$

F.T

- Exponential fn
- Trigometric
- Constant
- Algebraic

Ex 2:- Find F.T of $f(x) = \begin{cases} \sin 2x & , x > 0 \\ 0 & , x < 0 \end{cases}$

$$\rightarrow \quad F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} \, dx = \int_{-\infty}^{0} + \int_0^\infty$$

$$= \int_0^\infty \sin 2x \, e^{-i\lambda x} \, dx$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$$

$a = -i\lambda$, $b = 2$

$$= \left[\frac{e^{-i\lambda x}}{i^2\lambda^2 + 4} (-i\lambda \sin 2x - 2\cos 2x) \right]_0^\infty$$

$$= 0 - \frac{1}{4-\lambda^2}(-2) = \frac{2}{4-\lambda^2}$$

$\begin{pmatrix} \sin 0 = 0 \\ \cos 0 = 1 \end{pmatrix}$

Ex 3:- Find F.T of $f(x) = \begin{cases} \cos 2x, & x > 0 \\ 0, & x < 0 \end{cases}$

$\rightarrow$ $F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx$

$= \int_{0}^{\infty} \cos 2x\, e^{-i\lambda x}\, dx$ $\qquad a = -i\lambda$, $b = 2$

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$= \left[\frac{e^{-i\lambda x}}{(-i\lambda)^2+4}(-i\lambda\cos 2x + 2\sin 2x)\right]_0^{\infty}$$

$$F(\lambda) = 0 - \frac{1}{4-\lambda^2}(-i\lambda) = \frac{i\lambda}{4-\lambda^2}$$

Ex 4:- Find F.T $f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$\rightarrow$ $F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx = \int_{-\infty}^{0} + \int_{0}^{\infty}$

$$= \int_0^{\infty} 1 \cdot e^{-i\lambda x}\, dx = \left[\frac{e^{-i\lambda x}}{-i\lambda}\right]_0^{\infty} = 0 + \frac{1}{i\lambda}$$

$$\therefore F\{f(x)\} = \frac{1}{i\lambda}$$

Formula:- $\int uv\, dx = uV_1 - u'V_2 + u''V_3 - u'''V_4 + \cdots$

dash - derivative

suffixes - integration

$u \rightarrow$ Algebraic

$v \rightarrow$ Trigonometric or exponential

Ex :- Find F.T of $f(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$F\{f(x)\} = \int_0^1 \underset{u}{x^2}\, \underset{v}{e^{-i\lambda x}}\, dx$$

$$= \left[x^2\left(\frac{e^{-i\lambda x}}{-i\lambda}\right) - (2x)\left(\frac{e^{-i\lambda x}}{(-i\lambda)^2}\right) + (2)\left(\frac{e^{-i\lambda x}}{(-i\lambda)^3}\right)\right]_0^1$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} \qquad \iiint e^{ax}\,dx\,dx\,dx = \frac{e^{ax}}{a^3}$$

$$\iint e^{ax}\,dx\,dx = \frac{e^{ax}}{a^2} \qquad i^2 = -1, \quad i^3 = i^2 \cdot i = -i$$

$$= \left[\left(\frac{e^{-i\lambda}}{-i\lambda} - 2\frac{e^{-i\lambda}}{-\lambda^2} + 2\frac{e^{-i\lambda}}{-i\lambda^3}\right) - \left(\frac{2}{-i\lambda^3}\right)\right]$$

$$= e^{-i\lambda}\left[\frac{1}{-i\lambda} + \frac{2}{\lambda^2} - \frac{2}{i\lambda^3}\right] + \frac{2}{i\lambda^3}$$

Defn 2 :- Inverse Fourier Transform :-

I.F.T of $F(\lambda)$ is defined as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda, \qquad -\infty < \lambda < \infty \quad \text{—— (2)}$$

This eqn (2) is also called as Fourier integral representation of $f(x)$.

Ex :- Find Fourier integral representation of $f(x) = \begin{cases} 1, & 0 \le x < \infty \\ 0, & x < 0 \end{cases}$

$\rightarrow$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} d\lambda \quad \text{—— ①}$$

Let us find $F(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx$

$$= \int_{0}^{\infty} 1 \cdot e^{-i\lambda x} dx = \left[\frac{e^{-i\lambda x}}{-i\lambda} \right]_0^{\infty}$$

$$F(\lambda) = \frac{1}{i\lambda}$$

$\therefore$ The Fourier integral representation of $f(x)$ is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\lambda} e^{i\lambda x} d\lambda$$

which is required answer.

Def^{n}3 :- Fourier Cosine transform (FCT)

FCT of $f(x)$ defined for $0 < x < \infty$, as

$$F_c(\lambda) = \int_0^{\infty} f(x) \cos\lambda x \, dx \quad \text{—— ③}$$

Note :- If $f(x)$ is an even function then we can find FCT

Even function : $f(-x) = f(x)$ for all x

ex : x^2, $\cos x$ etc

Ex:- Find F.C.T of $f(x) = e^{-5x}$, $x > 0$

$\rightarrow$ $$F_c(\lambda) = \int_0^\infty f(x)\cos\lambda x \, dx$$

$$= \int_0^\infty e^{-5x}\cos\lambda x \, dx \qquad a = -5,\ b = \lambda$$

$$= \left[\frac{e^{-5x}}{25+\lambda^2}\left(-5\cos\lambda x + \lambda\sin\lambda x\right)\right]_0^\infty$$

$$= 0 - \frac{1}{25+\lambda^2}(-5) = \frac{5}{25+\lambda^2}$$

$$\int e^{ax}\cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)$$

Ex:- Find F.C.T of $f(x) = \begin{cases} x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

$\rightarrow$ $$F_c(\lambda) = \int_0^\infty f(x)\cos\lambda x \, dx$$

$$= \int_0^1 x^2\cos\lambda x \, dx$$

$$= \left[x^2\left(\frac{\sin\lambda x}{\lambda}\right) - (2x)\left(-\frac{\cos\lambda x}{\lambda^2}\right) + (2)\left(-\frac{\sin\lambda x}{\lambda^3}\right)\right]_0^1$$

$$= \left(\frac{\sin\lambda}{\lambda} + \frac{2\cos\lambda}{\lambda^2} - \frac{2\sin\lambda}{\lambda^3}\right) - 0$$

$$F_c(\lambda) = \frac{\sin\lambda}{\lambda} + \frac{2\cos\lambda}{\lambda^2} - \frac{2\sin\lambda}{\lambda^3}$$

Def^n 4:- Fourier Sine transform (FST), $0<x<\infty$

$$F_s(\lambda) = \int_0^\infty f(x)\sin\lambda x\, dx \quad \text{--- (4)}$$

Note:- If $f(x)$ is an odd fn ie $f(-x) = -f(x)$ then we find $F_s(\lambda)$

Ex:- Find the FST of $f(x) = e^{-4x}$, $x>0$

$\rightarrow$ $$F_s(\lambda) = \int_0^\infty f(x)\sin\lambda x\, dx$$

$$= \int_0^\infty e^{-4x}\sin\lambda x\, dx$$

$$= \left[\frac{e^{-4x}}{16+\lambda^2}\left(-4\sin\lambda x - \lambda\cos\lambda x\right)\right]_0^\infty$$

$$= 0 - \frac{1}{16+\lambda^2}(-\lambda) = \frac{\lambda}{16+\lambda^2}$$

Ex:- Find FST of $f(x) = \begin{cases} x^2, & 0\le x<1 \\ 0 & \text{otherwise} \end{cases}$

$\rightarrow$ $$F_s(\lambda) = \int_0^\infty f(x)\sin\lambda x\, dx$$

$$= \int_0^1 x^2\sin\lambda x\, dx$$

$$= \left[x^2\left(\frac{-\cos\lambda x}{\lambda}\right) - (2x)\left(-\frac{\sin\lambda x}{\lambda^2}\right) + (2)\left(\frac{\cos\lambda x}{\lambda^3}\right)\right]_0^1$$

$$= \left(-\frac{\cos\lambda}{\lambda} + \frac{2\sin\lambda}{\lambda^2} + \frac{2\cos\lambda}{\lambda^3}\right) - \left(\frac{2}{\lambda^3}\right)$$