Z-transform

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$$\boxed{\{f(k)\}} \xrightarrow{Z\text{-trans.}} \boxed{F(z)}$$

Discrete Valued f^n

Def^n:- $$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = F(z)$$

Z - transform of some std. sequences

1) $$Z\{a^k\}_{k\geq 0} = \frac{z}{z-a}, \quad |z|>|a|$$

Pf:-

$$Z\{a^k\}_{k\geq 0} = \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= 1+(az^{-1})+(az^{-1})^2+(az^{-1})^3+\cdots$$

W.K.t $1+x+x^2+\cdots = \frac{1}{1-x}, \quad |x|<1$

$$= \frac{1}{1-az^{-1}}, \quad |az^{-1}|<1$$

$$= \frac{1}{1-\frac{a}{z}} \qquad |a|<|z|$$

$$Z\{a^k\}_{k\geq 0} = \frac{z}{z-a}, \quad |z|>|a|$$

2) $$Z\{a^k\}_{k<0} = \frac{z}{a-z}, \quad |z|<|a|$$

$\rightarrow$ $$Z\{a^k\}_{k<0} = \sum_{k=-1}^{-\infty} a^k z^{-k}$$

Put $k=-m$

$k=-1, \; m=1$

$k=-\infty, \; m=\infty$

$$= \sum_{m=1}^{\infty} a^{-m} z^{m}$$

$$= \bar{a}^{1}z + (\bar{a}^{1}z)^{2} + (\bar{a}^{1}z)^{3} + \cdots$$

$$= \bar{a}^{1}z\left[1 + \bar{a}^{1}z + (\bar{a}^{1}z)^{2} + \cdots\right]$$

$$= \bar{a}^{1}z\left[\frac{1}{1-\bar{a}^{1}z}\right], \quad |\bar{a}^{1}z| < 1$$

$$Z\{a^{k}\}_{k<0} = \frac{z}{a-z}, \quad |z| < |a|$$

③ $$Z\left\{\frac{a^{k}}{k!}\right\}_{k\geq 0} = \sum_{k=0}^{\infty} \frac{a^{k}}{k!} \bar{z}^{k}$$

$$= 1 + \frac{(a\bar{z}^{1})}{1!} + \frac{(a\bar{z}^{1})^{2}}{2!} + \frac{(a\bar{z}^{1})^{3}}{3!} + \cdots$$

$$= e^{a\bar{z}^{1}} \quad \text{for all } z$$

$$\therefore Z\left\{\frac{a^{k}}{k!}\right\}_{k\geq 0} = e^{a/z} \quad \text{for all } z.$$

④ $$Z\{\cos\alpha k\}_{k\geq 0} = \frac{z(z-\cos\alpha)}{z^{2}-2z\cos\alpha+1}, \quad |z| > 1$$

⑤ $$Z\{\sin\alpha k\}_{k\geq 0} = \frac{z\sin\alpha}{z^{2}-2z\cos\alpha+1}, \quad |z| > 1$$

⑥ $$Z\{\cosh\alpha k\}_{k\geq 0} = \frac{z(z-\cosh\alpha)}{z^{2}-2z\cosh\alpha+1}, \quad |z| > \max\{|e^{\alpha}|, |\bar{e}^{\alpha}|\}$$

⑦ $$Z\{\sinh\alpha k\}_{k\geq 0} = \frac{z\sinh\alpha}{z^{2}-2z\cosh\alpha+1}, \quad |z| > \max\{|e^{\alpha}|, |\bar{e}^{\alpha}|\}$$

Properties of Z-transform :-

1) Linearity Property :-

$$Z\{a f(k) + b g(k)\} = a Z\{f(k)\} + b Z\{g(k)\} = a F(z) + b G(z)$$

Example 1) Find $Z\{\sin(2k+3)\}$

$\rightarrow$ $\sin(2k+3) = \sin 2k \cos 3 + \cos 2k \sin 3$

$\therefore Z\{\sin(2k+3)\} = \cos 3 \, Z\{\sin 2k\} + \sin 3 \, Z\{\cos 2k\}$

$$= \cos 3 \frac{z \sin 2}{z^2 - 2z\cos 2 + 1} + \sin 3 \frac{z(z - \cos 2)}{z^2 - 2z\cos 2 + 1}$$

$$= \frac{\cos 3 \, z \sin 2 + z \sin 3 (z - \cos 2)}{z^2 - 2z\cos 2 + 1}$$

2) $Z\{\cos(2k+3)\} = Z\{\cos 2k \cos 3 - \sin 2k \sin 3\}$

$$= \cos 3 \, Z\{\cos 2k\} - \sin 3 \, Z\{\sin 2k\}$$

$$= \cos 3 \frac{z(z - \cos 2)}{z^2 - 2z\cos 2 + 1} - \sin 3 \frac{z \sin 2}{z^2 - 2z\cos 2 + 1}$$

3) Find $Z\{\cos(7k+2)\}$, $k \geq 0$

$\rightarrow$ $Z\{\cos(7k+2)\} = Z\{\cos 7k \cos 2 - \sin 7k \sin 2\}$

$$= \cos 2 \, Z\{\cos 7k\} - \sin 2 \, Z\{\sin 7k\}$$

$$= \cos 2 \frac{z(z-\cos 7)}{z^2-2z\cos 7+1} - \sin 2 \frac{z\sin 7}{z^2-2z\cos 7+1}, \quad |z|>1$$

4) $Z\{\sin(\frac{k\pi}{2}+\alpha)\}, \; k \geq 0$

$\rightarrow$ $$Z\{\sin(\frac{k\pi}{2}+\alpha)\} = Z\{\sin\frac{k\pi}{2}\cdot\cos\alpha + \cos\frac{k\pi}{2}\sin\alpha\}$$

$$= \cos\alpha\, Z\{\sin\frac{k\pi}{2}\} + \sin\alpha\, Z\{\cos\frac{k\pi}{2}\}$$

$$= \cos\alpha\left[\frac{z\sin\pi/2}{z^2-2z\cos\pi/2+1}\right] + \sin\alpha\left[\frac{z(z-\cos\pi/2)}{z^2-2z\cos\pi/2+1}\right]$$

$\cos\pi/2 = 0$

$\sin\pi/2 = 1$

$$= \cos\alpha \frac{z}{z^2+1} + \sin\alpha \frac{z^2}{z^2+1}$$

$$= \frac{z\cos\alpha + z^2\sin\alpha}{z^2+1}$$

Ex :- $Z\{\sinh\frac{k\pi}{2}\}, \; k \geq 0$

$\rightarrow$ $$Z\{\sinh\alpha k\} = \frac{z\sinh\alpha}{z^2-2z\cosh\alpha+1}$$

$\alpha = \pi/2$, $$Z\{\sinh\frac{k\pi}{2}\} = \frac{z\sinh\pi/2}{z^2-2z\cosh\pi/2+1} = \frac{z\sinh\pi/2}{z^2-2z\cosh\pi/2+1}$$

$$|z| > \max\{|e^{\pi/2}|, |e^{-\pi/2}|\}$$

Ex :- $$Z\{\cosh\frac{k\pi}{2}\} = \frac{z(z-\cosh\pi/2)}{z^2-2z\cosh\pi/2+1}, \quad |z|>\max\{|e^{\pi/2}|, |e^{-\pi/2}|\}$$

Ex:- $Z\left\{\frac{3^k}{k!}\right\}_{k\geq 0} = \sum_{k=0}^{\infty} \frac{3^k}{k!} z^{-k} = e^{3/z}$ for all z

Ex:- $Z\left\{2^k + \left(\frac{1}{2}\right)^k\right\}_{k\geq 0} = Z\{2^k\} + Z\left\{\left(\frac{1}{2}\right)^k\right\}$

$$= \frac{z}{z-2} + \frac{z}{z-1/2}$$

Ex: $Z\{e^{k\alpha}\}_{k\geq 0}$ Here $a = e^{\alpha}$

$$= \frac{z}{z-e^{\alpha}}, \quad |z| > |e^{\alpha}|$$

Ex:- $Z\left\{\cos\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)\right\}$ $k \geq 0$

Homework

$Z\left\{\sin\left(\frac{k\pi}{2} + \alpha\right)\right\}$ $k \geq 0$

② Change of scale :-

If $Z\{f(k)\} = F(z)$ then $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$

Ex:- Find $Z\{f(k)\}$ if

i) $f(k) = 2^k \cos(3k+2)$, $k \geq 0$

ii) $f(k) = 4^k \sin(2k+3)$, $k \geq 0$

iii) $f(k) = 3^k \sinh \alpha k$, $k \geq 0$

iv) $f(k) = 2^k \cosh \alpha k$, $k \geq 0$

Sol^n 1) $Z\{2^k \cos(3k+2)\}$

First find $Z\{\cos(3k+2)\}$

$$Z\{\cos(3k+2)\} = Z\{\cos 3k \cos 2 - \sin 3k \sin 2\}$$

$$= \cos 2 \, Z\{\cos 3k\} - \sin 2 \, Z\{\sin 3k\}$$

$$= \cos 2 \, \frac{z(z-\cos 3)}{z^2 - 2z\cos 3 + 1} - \sin 2 \, \frac{z \sin 3}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z\cos 2(z-\cos 3) - \sin 2 \sin 3 \cdot z}{z^2 - 2z\cos 3 + 1}$$

By using change of scale property,

Replace z by $z/2$

$$\therefore Z\{2^k \cos(3k+2)\} = \frac{(z/2)\cos 2\,(z/2 - \cos 3) - (z/2)\sin 2 \sin 3}{(z/2)^2 - 2(z/2)\cos 3 + 1}$$

2) $Z\{4^k \sin(2k+3)\}, \; k \geq 0$

$\rightarrow$ $$Z\{\sin(2k+3)\} = Z\{\sin 2k \cos 3 + \cos 2k \sin 3\}$$

$$= \cos 3 \, Z\{\sin 2k\} + \sin 3 \, Z\{\cos 2k\}$$

$$= \cos 3 \, \frac{z \sin 2}{z^2 - 2z\cos 2 + 1} + \sin 3 \, \frac{z(z-\cos 2)}{z^2 - 2z\cos 2 + 1}$$

$$\therefore Z\{4^k \sin(2k+3)\} = \frac{(z/4)\sin 2 \cos 3 + (z/4)(z/4 - \cos 2)\sin 3}{(z/4)^2 - 2(z/4)\cos 2 + 1}$$

3) If $Z\{f(k)\} = F(z)$ then $Z\{e^{-ak} f(k)\} = F(e^{a}z)$

Ex:- $Z\{e^{-ak} \cos bk\}$, $k \geq 0$

$\rightarrow$ $Z\{\cos bk\} = \dfrac{z(z-\cos b)}{z^2 - 2z\cos b + 1}$

$\therefore Z\{e^{-ak} \cos bk\} = \dfrac{(e^{a}z)(e^{a}z - \cos b)}{(e^{a}z)^2 - 2(e^{a}z)\cos b + 1}$

Ex:- $Z\{e^{-3k} \cos 4k\}$ $k \geq 0$

$\rightarrow$ $Z\{\cos 4k\} = \dfrac{z(z-\cos 4)}{z^2 - 2z\cos 4 + 1}$

$\therefore Z\{e^{-3k} \cos 4k\} = \dfrac{(e^{3}z)(e^{3}z - \cos 4)}{(e^{3}z)^2 - 2(e^{3}z)\cos 4 + 1}$

H.W (1) $Z\{e^{-ak} \sin bk\}$ $k \geq 0$

(2) $Z\{e^{-4k} \cosh \alpha k\}$ $k \geq 0$

(3) $Z\{e^{-3k} \sin 4k\}$ $k \geq 0$

(4) <u>Multiplication by k</u>

If $Z\{f(k)\} = F(z)$ then $Z\{k f(k)\} = -z \dfrac{d}{dz} F(z)$

IIrly $Z\{k^2 f(k)\} = (-z)^2 \dfrac{d^2}{dz^2} F(z)$

Ex:- $Z\{K\,5^k\}_{k\geq 0}$

$\rightarrow$ $Z\{5^k\} = \dfrac{z}{z-5}$, $k \geq 0$ $\qquad \dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}$

$$\therefore Z\{K\,5^k\} = -z\,\frac{d}{dz}\left[\frac{z}{z-5}\right]$$

$$= -z\left[\frac{(z-5)\cdot 1 - z(1)}{(z-5)^2}\right] = -z\left[\frac{-5}{(z-5)^2}\right]$$

$$= \frac{5z}{(z-5)^2}$$

Ex:- $Z\{k\,4^k\}_{k\geq 0} = \dfrac{4z}{(z-4)^2}$

Ex:- $Z\{K\}_{k\geq 0}$

$\rightarrow$ $K = K\cdot 1^k$

$$Z\{1^k\}_{k\geq 0} = \frac{z}{z-1}$$

$$Z\{K\} = -z\,\frac{d}{dz}\left[\frac{z}{z-1}\right]$$

$$= -z\left[\frac{(z-1)\cdot 1 - z(1)}{(z-1)^2}\right]$$

$$Z\{k\} = \frac{z}{(z-1)^2}$$

Ex:- $Z\{k\,e^{-ak}\}_{k\geq 0}$

$\rightarrow$ $Z\{k\} = \dfrac{z}{(z-1)^2}$ $\quad \therefore Z\{k\,e^{-ak}\} = \dfrac{e^a z}{(e^a z-1)^2}$

Ex :- $Z\{(k+1)a^k\}_{k\geq 0}$

$$= Z\{ka^k\}_{k\geq 0} + Z\{a^k\}_{k\geq 0}$$

$$= \frac{az}{(z-a)^2} + \frac{z}{z-a}$$

(5) Division by k

If $Z\{f(k)\} = F(z)$ then $Z\left\{\frac{f(k)}{k}\right\} = \int_z^\infty \frac{F(z)}{z}\,dz$

Ex :- Find $Z\left\{\frac{\sin ak}{k}\right\}$, $k>0$.

$\rightarrow$ $Z\{\sin ak\} = \dfrac{z\sin a}{z^2-2z\cos a+1}$

$$Z\left\{\frac{\sin ak}{k}\right\} = \int_z^\infty \frac{\cancel{z}\sin a}{z^2-2z\cos a+1} \times \frac{1}{\cancel{z}}\,dz$$

$$= \sin a \int_z^\infty \frac{dz}{z^2-2z\cos a+1}$$

$$= \sin a \int_z^\infty \frac{dz}{(\underbrace{z^2-2z\cos a+\cos^2 a}-\cos^2 a+1)}$$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$$= \sin a \int_z^\infty \frac{dz}{(z-\cos a)^2+\sin^2 a}$$

$$= \cancel{\sin a} \times \frac{1}{\cancel{\sin a}} \tan^{-1}\left[\frac{z-\cos a}{\sin a}\right]_z^\infty$$

$$= \pi/2 - \tan^{-1}\left[\frac{z-\cos a}{\sin a}\right] = \cot^{-1}\left(\frac{z-\cos a}{\sin a}\right)$$