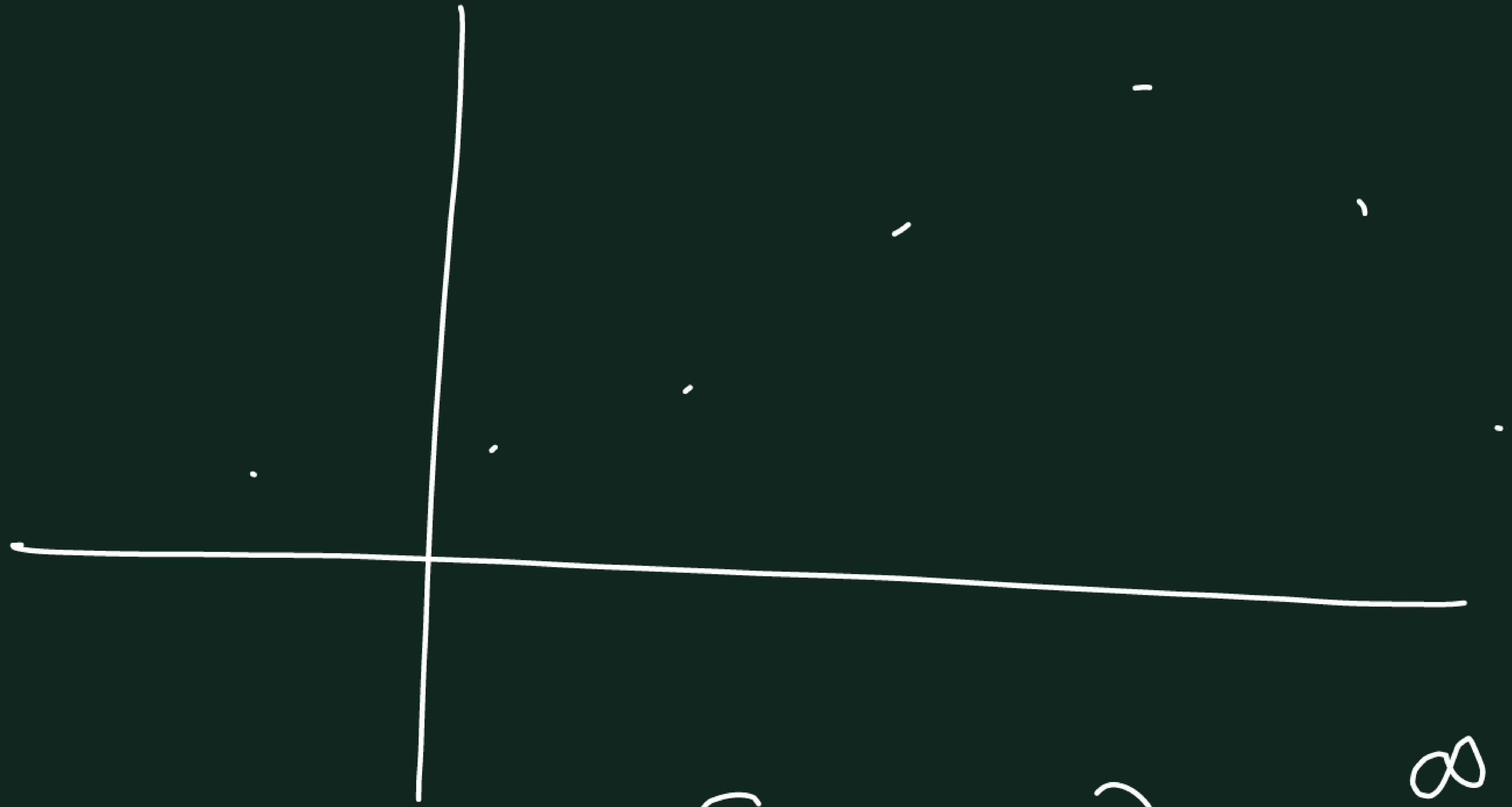
Z transform

$$1 + z + z^2 + \cdots = \frac{1}{1-z}, \quad |z|<1$$

Discrete valued f^n $\{f(k)\}_{k=-\infty}^{\infty}$

ROC



$$\boxed{\{f(k)\}} \xrightarrow{\text{Z-transform}} \boxed{F(z)}$$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = F(z)$$

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx$$

① Z transform of some std. sequences.

① $f(k) = a^k, \; k \ge 0$.

$$Z\{a^k\}_{k\ge 0} = \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= 1 + (az^{-1}) + (az^{-1})^2 + \cdots$$

$$= \frac{1}{1-az^{-1}}, \quad |az^{-1}| < 1$$

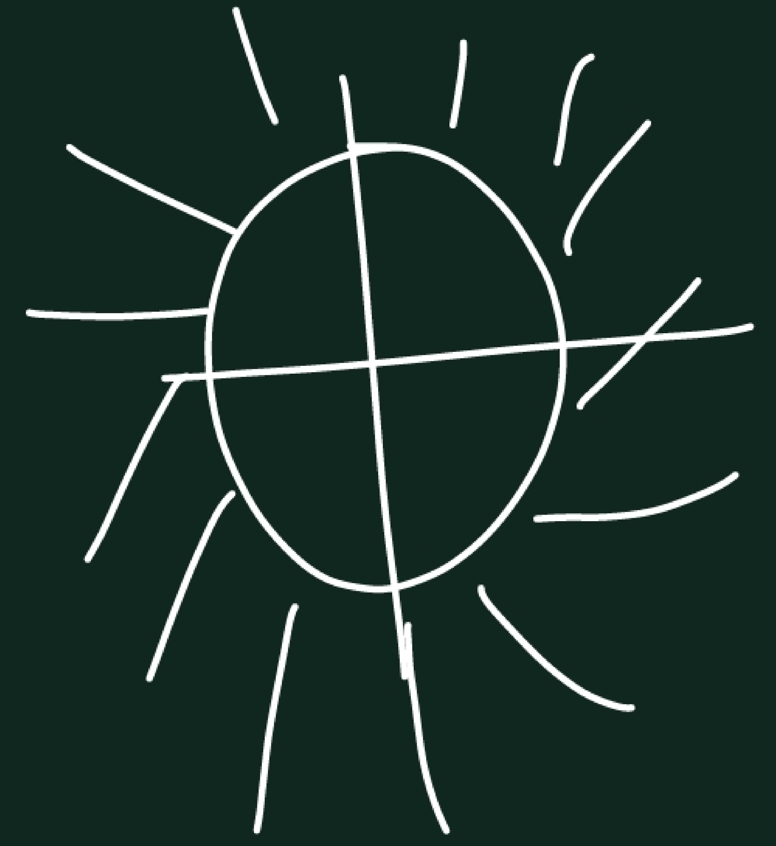
$$\boxed{Z\{a^k\}_{k\ge 0} = \frac{z}{z-a} \qquad |z| > |a|}$$

$1 + x + x^2 + \cdots = \frac{1}{1-x}, \; |x| < 1$

$z = x + iy$

$|z| = \sqrt{x^2 + y^2}$

$|z| > |a| \Rightarrow x^2 + y^2 > a^2$



② $f(k) = a^k, \quad k < 0$

$$Z\{a^k\}_{k<0} = \sum_{k=-1}^{-\infty} a^k z^k$$

$$\boxed{\begin{aligned} &\text{Put } k = -m \\ &k=-1, \; m = 1 \\ &k = -\infty, \; m = \infty \end{aligned}}$$

$$= \sum_{m=1}^{\infty} a^{-m} z^m$$

$$= a^{-1}z + (a^{-1}z)^2 + (a^{-1}z)^3 + \cdots$$

$$= a^{-1}z \left[1 + a^{-1}z + (a^{-1}z)^2 + \cdots\right]$$

$$= a^{-1}z \left[\frac{1}{1 - a^{-1}z}\right], \quad |a^{-1}z| < 1$$

$$\boxed{Z\{a^k\}_{k<0} = \frac{z}{a - z}, \quad |z| < |a|}$$

Find $Z\{f(k)\}$ where $f(k) = 4^k, k \geq 0$.

$\rightarrow Z\{4^k\}_{k\geq 0} = \dfrac{z}{z-4}, \; |z| > 4$.

Find $Z\{f(k)\}$ where $f(k) = \begin{cases} 3^k, & k \geq 0 \\ 4^k, & k < 0 \end{cases}$

$\rightarrow Z\{f(k)\} = Z\{3^k\}_{k\geq 0} + Z\{4^k\}_{k<0}$

$= \dfrac{z}{z-3} + \dfrac{z}{4-z} \quad 3 < |z| < 4$

③ $f(k) = \frac{a^k}{k!}, \; k \geq 0$

$$Z\left\{\frac{a^k}{k!}\right\}_{k\geq 0} = \sum_{k=0}^{\infty} \frac{a^k}{k!} z^{-k}$$

$$\boxed{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x}$$

$$= 1 + \frac{az^{-1}}{1!} + \frac{(az^{-1})^2}{2!} + \frac{(az^{-1})^3}{3!} + \cdots$$

$$= e^{az^{-1}} \quad \text{for all } z$$

$$= e^{a/z}$$

$$\cos\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2}$$

④ $f(k) = \sin\alpha k, \quad k \geq 0$

$$Z\{\sin\alpha k\}_{k\geq 0} = \frac{z\sin\alpha}{z^2-2z\cos\alpha+1}, \quad |z|>1$$

<u>Proof</u>: $\sin\alpha k = \dfrac{e^{i\alpha k}-e^{-i\alpha k}}{2i}$

$$\therefore Z\{\sin\alpha k\} = \frac{1}{2i} Z\{e^{i\alpha k}-e^{-i\alpha k}\}$$

$$= \frac{1}{2i}\left[Z\{e^{i\alpha k}\} - Z\{e^{-i\alpha k}\}\right]$$

$$= \frac{1}{2i}\left[\frac{z}{z-e^{i\alpha}} - \frac{z}{z-e^{-i\alpha}}\right], \quad |z|>|e^{i\alpha}|, \ |z|>|e^{-i\alpha}|$$

$$= \frac{1}{2i} z\left[\frac{z-e^{-i\alpha}-z+e^{i\alpha}}{(z-e^{i\alpha})(z-e^{-i\alpha})}\right]$$

$$= \frac{1}{\cancel{2i}}\,\frac{z\times \cancel{2i}\sin\alpha}{z^2-(e^{i\alpha}+e^{-i\alpha})z+1}$$

$$= \frac{z\sin\alpha}{z^2-2z\cos\alpha+1}$$

Ex:- Find $Z\{\sin 4k\}_{k\geq 0}$.

$$\longrightarrow Z\{\sin 4k\}_{k\geq 0} = \frac{z\sin 4}{z^2 - 2z\cos 4 + 1}, \quad |z|>1$$

(5) $Z\{\cos\alpha k\}_{k\geq 0} = \dfrac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}, \quad |z|>1$

eg: $Z\{\cos 4k\}_{k\geq 0} = \dfrac{z(z-\cos 4)}{z^2-2z\cos 4+1}, \quad |z|>1$

(6) $Z\{\cosh\alpha k\} = \dfrac{z(z-\cosh\alpha)}{z^2-2z\cosh\alpha+1}, \quad |z|>\max\{|e^{\alpha}|, |e^{-\alpha}|\}$

(7) $Z\{\sinh\alpha k\} = \dfrac{z\sinh\alpha}{z^2-2z\cosh\alpha+1}, \quad |z|>\max\{|e^{\alpha}|, |e^{-\alpha}|\}$