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Class:- COMP I   Year:- SE

M-3 Assignment

Q1. Example on variation of parameters

1. $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$

Soln: Let $D = \dfrac{d}{dx}$

$\therefore D^2y + 4y = \tan 2x$

$\therefore D^2 + 4 = 0$ is the AE

$\therefore D = \pm 2i$

if $D = \alpha \pm \beta i$

then $CF = e^{\alpha x}[C_1 \cos\beta x + C_2 \sin\beta x]$

$CF = C_1 \cos 2x + C_2 \sin 2x$

$y_p = \dfrac{1}{\phi(D)} f(x)$

$= \dfrac{\tan 2x}{D^2+4}$

$\therefore y_1 = \cos 2x \;;\; y_2 = \sin 2x$

Let $PI = u\cos 2x + v \sin 2x$

$y_1' = -2\sin 2x \;;\; y_2' = 2\cos 2x$

$M = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$

$= 2\cos^2 2x + 2\sin^2 2x$

$= 2$

$u = -\displaystyle\int \frac{y_2 f(x)}{M}\, dx$

$= -\displaystyle\int \frac{\sin 2x \tan 2x}{2}\, dx$

$= -\displaystyle\int \frac{\sin^2 2x}{2\cos 2x}\, dx$

$= -\dfrac{1}{2}\displaystyle\int \sec 2x\, dx + \dfrac{1}{2}\int \cos 2x\, dx$

$$= \frac{\sin 2x}{4} - \frac{\log(\sec 2x + \tan 2x)}{4}$$

$$v = \int \frac{y_1 f(x)}{W} dx$$

$$= \int \frac{\cos 2x \tan 2x}{2} dx$$

$$= \int \frac{\sin 2x}{2} dx$$

$$= \frac{-\cos 2x}{4}$$

$$\therefore y_p = \left[\frac{\sin 2x}{4} - \frac{\log(\sec 2x + \tan 2x)}{4}\right] \cos 2x + \left[-\frac{\cos 2x}{4}\right] \sin 2x$$

$$\therefore y_p = \frac{-\log(\sec 2x + \tan 2x) \times \cos 2x}{4}$$

$\therefore$ General solution,

$$y = y_c + y_p$$

$$y = C_1 \cos 2x + C_2 \sin 2x - \frac{\log(\sec 2x + \tan 2x) \times \cos 2x}{4}$$

2. $\frac{d^2y}{dx^2} + y = x \sin x$

Soln: Let $D = \frac{d}{dx}$

$\therefore D^2y + y = x \sin x$

$\therefore D^2 + 1 = 0$ is the AE

$\therefore D = \pm i$

if $D = \alpha \pm \beta i$

then $CF = e^{\alpha x}[C_1 \cos \beta x + C_2 \sin \beta x]$

$CF = C_1 \cos x + C_2 \sin x$

$\therefore y_1 = \cos x \qquad y_2 = \sin x$

$y_1' = -\sin x \qquad y_2' = \cos x$

$$\therefore M = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$M = 1$$

$$u = -\int \frac{y_2 f(x)}{M}\, dx$$

$$= -\int \frac{\sin x \times x \sin x}{1}\, dx$$

$$= -\int x \left[\frac{1 - \cos 2x}{2}\right] dx$$

$$= -\int \frac{x}{2}\, dx + \frac{1}{2}\int x \cos 2x\, dx$$

$$u = \left[\frac{x \sin 2x}{4} + \frac{\cos 2x}{8}\right] - \frac{x^2}{4}$$

$$V = \int \frac{y_1 f(x)}{M}\, dx$$

$$= \int \cos x \times x \sin x\, dx$$

$$= \frac{1}{2}\int x \sin 2x\, dx$$

$$= \frac{1}{2}\left[+\frac{\sin 2x}{4} - \frac{x \cos 2x}{2}\right]$$

$$\cancel{= \frac{\cos 2x}{8} - \frac{x \sin 2x}{4}}$$

$$V = \frac{\sin 2x}{8} - \frac{x \cos 2x}{4}$$

$$\therefore\ y_p = \left[\frac{x \sin 2x}{4} + \frac{\cos 2x}{8} - \frac{x^2}{4}\right] \cos x$$

$$+ \left[\frac{\sin 2x}{8} - \frac{x \cos 2x}{4}\right] \sin x$$

$\therefore$ General solution

$y = y_c + y_p$

3. $(D^2+3D+2)y = \sin e^x$

soln:

$\therefore D^3+3D+2=0$ is the AE

$\therefore (D+1)(D+2)=0$

$D=-1,-2$

if $D = a_1, a_2$

then $CF = C_1 e^{a_1 x} + C_2 e^{a_2 x}$

$CF = C_1 e^{-x} + C_2 e^{-2x}$

~~y_p~~ $y_1 = e^{-x}$; $y_2 = e^{-2x}$

$y_1' = -e^{-x}$; $y_2' = -2e^{-2x}$

$$M = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$= -2e^{-3x} + e^{-3x}$

$M = -e^{-3x}$

$$u = -\int \frac{y_2 f(x)}{M}\,dx$$

$$= \int \frac{e^{-2x}\sin(e^x)}{e^{-3x}}\,dx$$

$$= \int e^x \sin(e^x)\,dx$$

let $e^x = t$

$e^x dx = dt$

$\therefore u = \int \sin t\,dt$

$= -\cos t$

$u = -\cos(e^x)$

$$v = \int \frac{y_1 f(x)}{M}\,dx$$

$$= -\int \frac{e^{-x}\sin(e^x)}{e^{-3x}}\,dx$$

$$= -\int e^x \times e^x \sin(e^x)\, dx$$

Let $e^x = t$

$e^x dx = dt$

$$= -\int t \sin t\, dt$$

$$= -[-t\cos t + \sin t]$$

$$= t\cos t - \sin t$$

$$v = e^x \cos(e^x) - \sin(e^x)$$

$$\therefore y_p = -\cos(e^x) \times e^{-x} + [e^x \cos(e^x) - \sin(e^x)] \times e^{-2x}$$

$\therefore$ General solution

$$y = y_c + y_p$$

4. $(D^2 - 2D)\, y = e^x \sin x$

Soln: $\therefore D^2 - 2D = 0$ is the AE

$D(D-2) = 0$

$\therefore D = 0, 2$

if $D = a_1, a_2$

then $CF = c_1 e^{a_1 x} + c_2 e^{a_2 x}$

$y_c = \cancel{\text{Ere}}\; c_1 + c_2 e^{2x}$

$\therefore PI = u \times 1 + v \times e^{2x}$

$y_1 = 1;\; y_2 = e^{2x}$

$y_1' = 0;\; y_2' = 2e^{2x}$

$$\therefore M = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix}$$

$$M = 2e^{2x}$$

$$u = -\int \frac{y_2 f(x)}{M} dx$$

$$= -\int \frac{e^{2x} \times e^x \sin x}{2e^{2x}} dx$$

$$= -\frac{1}{2} \int e^x \sin x\, dx$$

$$u = -\frac{1}{4} \times e^x [\sin x - \cos x]$$

$$v = \int \frac{y_1 f(x)}{M} dx$$

$$= \int \frac{e^{x} \sin x}{2e^{2x}} dx$$

$$= \frac{1}{2}\int e^{-x} \sin x \, dx$$

$$v = \frac{1}{4} e^{-x}[-\sin x - \cos x]$$

$$\therefore y_c = -\frac{e^x}{4}(\sin x - \cos x) - \frac{e^x}{4}(\sin x + \cos x)$$

$$y_c = -\frac{e^x \sin x}{2}$$

$\therefore$ General solution

$$y = y_c + y_p$$

5. $(D^2-1)y = \frac{2}{1+e^x}$

Soln:

$\therefore D^2-1=0$ is the AE

$\therefore D = \pm 1$

if $D = a_1, a_2$

then $CF = C_1 e^{a_1 x} + C_2 e^{a_2 x}$

$= C_1 e^{x} + C_2 e^{-x}$

$\therefore y_1 = e^x \; ; \; y_2 = e^{-x}$

$y_1' = e^x \; ; \; y_2' = -e^{-x}$

$$M = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$= \cancel{e^{2x}} -1-1 = -2$$

$$u = -\int \frac{y_2 f(x)}{M} dx$$

$$= \frac{1}{2}\int \frac{e^{-x} \times 2}{1+e^x} dx$$

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$$= \int \frac{1}{e^x(1+e^x)}\, dx$$

$$= \int \frac{(1+e^x)-e^x}{e^x(1+e^x)}\, dx$$

$$= \int \frac{1}{e^x}\, dx - \int \frac{1}{(1+e^x)}\, dx$$

$$= -e^{-x} + \int \frac{-e^{-x}}{e^{-x}(1+e^x)}\, dx$$

$$= -e^{-x} + \cancel{\log e} \int \frac{-e^{-x}}{e^{-x}+1}\, dx$$

$$u = -e^{-x} + \log(1+e^{-x})$$

$$v = \int \frac{y_1 f(x)}{M}\, dx$$

$$= -\int \frac{e^x}{x} \times \frac{x}{(1+e^x)}\, dx$$

$$= -\int \frac{e^x}{1+e^x}\, dx$$

$$v = -\log(1+e^x)$$

~~∴ General soluti~~

$$y_c = [\log(1+e^{-x}) - e^{-x}]e^x - e^{-x}\log(1+e^x)$$

∴ General solution

$$y = y_c + y_p$$

Q2. Examples on Cauchy's and Legendre's Linear Differential Equations

1. $(x^2D^2 - 4xD + 6)y = x^5$

Soln:

put $x = e^z$

$z = \log x$

$x\frac{dy}{dx} = Dy$

$x^2 \frac{d^2y}{dx^2} = D(D-1)y$

where $D = \frac{d}{dz}$

$\therefore$ $D(D-1)y - 4Dy + 6y = e^{5z}$

$D^2y - 5Dy + 6y = e^{5z}$

$\therefore$ $(D^2 - 5D + 6) = 0$ is the AE

$\therefore D = 2, 3$

if $D = 2, 3$

then $CF = C_1 e^{a_1 z} + C_2 e^{a_2 z}$

$\therefore CF = C_1 e^{2z} + C_2 e^{3z}$

$y_p = \frac{1}{f(D)} f(z)$

$= \frac{1}{D^2 - 5D + 6} e^{5z}$

$D \to 5$

$= \frac{1}{25 - 25 + 6} e^{5z}$

$y_p = \frac{e^{5z}}{6}$

$\therefore$ General solution

$y = y_c + y_p$

$= C_1 e^{2z} + C_2 e^{3z} + \frac{e^{5z}}{6}$

$y = C_1 x^2 + C_2 x^3 + \frac{x^5}{6}$

2. $(x^3 D^2 + 3x^2 D + x)y = \sin(\log x)$

Soln: ~~put $x = e^z$~~

$(x^2 D^2 + 3x D + 1)y = x^{-1} \sin(\log x)$

put $x = e^z$

$z = \log x$

$\cancel{x^2 D^2}$ $\quad x^2 \frac{d^2y}{dx^2} = D(D-1)y$

$x \frac{dy}{dx} = Dy$

$\therefore D = \frac{d}{dz}$

$D(D-1)y + 3Dy + y = e^{-z} \sin z$

$D^2 y + 2Dy + y = e^{-z} \sin z$

$\therefore (D^2 + 2D + 1) = 0$ is the AE

$\therefore D = -1, -1$

if $D = a, a$

then $CF = [C_1 + C_2 z] e^{az}$

$CF = [C_1 + C_2 z] e^{-z}$

$y_p = \frac{1}{f(D)} f(z)$

$= \frac{1}{(D+1)^2} e^{-z} \sin z$

$D \to D-1$

$= e^{-z} \left[\frac{1}{D^2} \sin z \right]$

$= -e^{-z} \frac{1}{D} \cos z$

$y_p = -e^{-z} \sin z$

$\therefore$ General solution

$y = y_c + y_p$

$= [C_1 + C_2 z] e^{-z} - e^{-z} \sin z$

$= [C_1 + C_2 \log x] x^{-1} - x^{-1} \sin(\log x)$

3. $(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x$

Soln:

Put $(2x+1) = e^z$

$z = \log(2x+1)$

$(2x+3)\frac{dy}{dx} = 2Dy$

$(2x+3)^2\frac{d^2y}{dx^2} = 4D(D-1)y$

$\therefore 4D(D-1)y - 4Dy - 12y = 6\left[\frac{e^z-1}{2}\right]$

$4D^2y - 8Dy - 12y = 3e^z - 3$

$D^2y - 2Dy - 3y = \frac{3e^z}{4} - \frac{3}{4}$

$\therefore D^2 - 2D - 3 = 0$ is the AE

$\therefore D^2 - 3D + D - 3 = 0$

$D(D-3) + 1(D-3) = 0$

$\therefore (D+1)(D-3) = 0$

$D = -1, 3$

if $D = a_1, a_2$

then $CF = c_1e^{a_1z} + c_2e^{a_2z}$

$CF = c_1e^{-z} + c_2e^{3z}$

$y_p = \frac{1}{\phi(D)} f(x)$

$= \frac{1}{D^2-2D-3} \times \frac{3e^z}{4} - \frac{1}{D^2-2D-3} \times \frac{3}{4}e^{0z}$

$D \to 1$ $\qquad D \to 0$

$= \frac{3}{4} \times \frac{e^z}{16-8-3} - \frac{3}{4} \times \frac{1}{-3}$

$y_p = \frac{3e^z}{20} + \frac{1}{4}$

$\therefore$ General solution

$y = y_c + y_p$

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$$y = C_1 e^{-z} + C_2 e^{3z} + \frac{3e^z}{20} + \frac{1}{4}$$

$$y = C_1 (2x+1)^{-1} + C_2 (2x+1)^3 + \frac{3(2x+1)}{20} + \frac{1}{4}$$

4. $(x+2)^2 \frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x+4$

Soln:

put $x+2 = e^z$

$z = \log(x+2)$

$(x+2)\frac{dy}{dx} = Dy$

$(x+2)^2 \frac{d^2y}{dx^2} = D(D-1)y$

$\therefore D(D-1)y - Dy + y = 3[e^z - 2] + 4$

$D^2y - 2Dy + y = 3e^z - 2$

$\therefore (D^2 - 2D + 1) = \cancel{3e^z - 2}\ 0$ is the AE

$\therefore D = 1, 1$

if $D = a, a$

then $CF = [C_1 + C_2 z]e^{az}$

$CF = [C_1 + C_2 z]e^z$

$y_p = \frac{1}{f(D)} f(z)$

$= \frac{1}{\underset{D \to 1}{D^2 - 2D + 1}} 3e^z - \frac{1}{\underset{D \to 0}{D^2 - 2D + 1}} 2e^{0z}$

$= \frac{3e^z}{1-2+1} - 2$

$= \frac{3z\, e^z}{\underset{D \to 1}{2D - 2}} - 2$

$y_p = \frac{3z^2 e^z}{2} - 2$

∴ General solution,

$$y = y_c + y_p$$

$$= [c_1 + c_2 z]e^z + \frac{3z^2e^z}{2} - 2$$

$$y = [c_1 + c_2 \cancel{(x+2)} \log(x+2)](x+2) + \frac{3[\log(x+2)]^2 (x+2)}{2} + 2$$

5. $(x^2D^2 - xD + 1)y = x\log x$

Soln: put $x = e^z$

$$z = \log x$$

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$\therefore D(D-1)y - Dy + y = e^z . z$$

$$D^2y - 2Dy + y = e^z \times z$$

$\therefore D^2 - 2D + 1 = \cancel{e^z \times z}\ 0$ is the AE

$$\therefore D = 1, 1$$

if $D = a, a$

then $CF = [c_1 + c_2 z]e^{az}$

$$CF = [c_1 + c_2 z]e^z$$

$$y_p = \frac{1}{\phi(D)} f(z)$$

$$= \frac{1}{\cancel{D^2 - 2D + 1}\ (D-1)^2} e^z \times z$$

$$D \to D+1$$

$$= e^z . \frac{1}{D^2} z$$

$$= e^z \frac{1}{D} \times \frac{z^2}{2}$$

$$= \frac{e^z z^3}{6}$$

$\therefore$ General solution,

$$y = y_c + y_p$$

$$= [c_1 + c_2 z] e^z + \frac{e^z z^3}{6}$$

$$y = [c_1 + c_2 \log x] x + \frac{x [\log x]^3}{6}$$