Inverse Z-transform

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = F(z) \quad \text{then} \quad Z^{-1}\{F(z)\} = \{f(k)\}$$

① $$Z^{-1}\left\{\frac{z}{z-a}\right\}, \ |z|>|a| = \{a^k\}_{k\geq 0}$$

② $$Z^{-1}\left\{\frac{z}{z-a}\right\}, \ |z|<|a| = \{-a^k\}_{k<0}$$

③ $$Z^{-1}\left\{\frac{1}{z-a}\right\}, \ |z|>|a| = \{a^{k-1}\}_{k\geq 0}$$

④ $$Z^{-1}\left\{\frac{1}{z-a}\right\}, \ |z|<|a| = \{-a^{k-1}\}_{k\leq 0}$$

⑤ $$Z^{-1}\left\{\frac{z}{(z-a)^2}\right\}, |z|>|a| = \{k a^{k-1}\}_{k\geq 0}$$

$$Z\{k a^k\}_{k\geq 0} = \frac{az}{(z-a)^2}, \ |z|>|a|$$

To find inverse Z-transform

1) Direct Method
2) Partial fraction Method $\longrightarrow$ $\boxed{\text{Imp}}$
3) Inversion integral formula

Example Find $Z^{-1}\left\{\frac{1}{(z-\frac{1}{2})(z-\frac{1}{3})}\right\}, \ \frac{1}{3}<|z|<\frac{1}{2}$

$\rightarrow$ $$\frac{1}{(z-\frac{1}{2})(z-\frac{1}{3})} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1}{3}}$$

$$1 = A\left(z-\tfrac{1}{3}\right) + B\left(z-\tfrac{1}{2}\right)$$

Put $z=\frac{1}{2}$ $\quad 1 = A\left(\frac{1}{2}-\frac{1}{3}\right) = A\left(\frac{1}{6}\right)$ $\quad A = 6$

$z=\frac{1}{3}$ $\quad 1 = B\left(\frac{1}{3}-\frac{1}{2}\right) = B\left(\frac{-1}{6}\right)$ $\quad B = -6$

$$\therefore \quad F(z) = \frac{6}{z-\frac{1}{2}} - \frac{6}{z-\frac{1}{3}}$$

$$\therefore \quad z^{-1}\{F(z)\} = 6\, z^{-1}\left\{\frac{1}{z-\frac{1}{2}}\right\}_{|z|<\frac{1}{2}} - 6\, z^{-1}\left\{\frac{1}{z-\frac{1}{3}}\right\}_{|z|>\frac{1}{3}}$$

$$= 6\left(\frac{-1}{2}\right)^{k-1} - 6\left(\frac{1}{3}\right)^{k-1}, \quad k \geq 0.$$

EX 2:- Find $z^{-1}\left\{\dfrac{1}{(z-3)(z-2)}\right\}$, $|z|<2$

$\rightarrow$

$$\frac{1}{(z-3)(z-2)} = \frac{A}{z-3} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-3)$$

Put $z=2$ $\quad B=-1$

$z=3$ $\quad A=1$

$$\therefore \quad z^{-1}\left\{\frac{1}{(z-3)(z-2)}\right\} = z^{-1}\left\{\frac{1}{z-3}\right\}_{|z|<3} - z^{-1}\left\{\frac{1}{z-2}\right\}_{|z|<2}$$

$$= -\{3^{k-1}\} - \{-2^{k-1}\}$$

$$= -3^{k-1} + 2^{k-1}, \quad k \leq 0$$

$F(z)$ $\{f(k)\}$

1. $\dfrac{z}{z-a}$, $|z|>|a|$ $\quad a^k$, $k\geq 0$

2. $\dfrac{z}{z-a}$, $|z|<|a|$ $\quad -a^{k-1}$, $k\leq 0$

3. $\dfrac{1}{z-a}$, $|z|>|a|$ $\quad a^{k-1}$, $k\geq 0$

4. $\dfrac{1}{z-a}$, $|z|<|a|$ $\quad -a^{k-1}$, $k\leq 0$

5. $\dfrac{z}{(z-a)^2}$, $|z|>|a|$ $\quad \{k a^{k-1}\}$ $k\geq 0$

Ex 3 Find $z^{-1}\left\{\dfrac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right\}$, $\frac{1}{5}<|z|<\frac{1}{4}$

$\rightarrow$ $F(z)=\dfrac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$

$$\frac{F(z)}{z}=\frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}=\frac{A}{z-\frac{1}{4}}+\frac{B}{z-\frac{1}{5}}$$

$$\therefore z=A\left(z-\tfrac{1}{5}\right)+B\left(z-\tfrac{1}{4}\right)$$

Put $z=\frac{1}{5}$ $\quad \therefore \frac{1}{5}=B\left(\frac{1}{5}-\frac{1}{4}\right)=B\left(\frac{-1}{20}\right)$

$$\therefore \boxed{B=-4}$$

Put $z=\frac{1}{4}$ $\quad \therefore \frac{1}{4}=A\left(\frac{1}{20}\right)$ $\quad \therefore \boxed{A=5}$

$$\frac{F(z)}{z}=\frac{5}{z-\frac{1}{4}}-\frac{4}{z-\frac{1}{5}}$$

$$\therefore \quad F(z) = \frac{5z}{z-\frac{1}{4}} - \frac{4z}{z-\frac{1}{5}}$$

$$\therefore \quad Z^{-1}\{F(z)\} = 5\, Z^{-1}\left\{\frac{z}{z-\frac{1}{4}}\right\} - 4\, Z^{-1}\left\{\frac{z}{z-\frac{1}{5}}\right\}$$

$$|z| > \frac{1}{4} \qquad\qquad |z| < \frac{1}{5}$$

$$= 5\left\{\left(\frac{1}{4}\right)^k\right\}_{k\geq 0} - 4\left\{-\left(\frac{1}{5}\right)^k\right\}_{k<0}$$

Solution of difference equation with const. coeff. using z-transform:-

Def^n:- A rel^n between $f(k+1), f(k+2), f(k+3), \ldots$ is called difference eq^n and $\{f(k)\}$ which satisfies the eq^n is called its solution.

Note:- $Z\{f(k)\} = F(z)$ $\qquad$ $\{f(k)\}$, $k \geq 0$ Causual seq.

$$Z\{f(k+1)\} = zF(z) - zf(0)$$

$$Z\{f(k+2)\} = z^2F(z) - z^2f(0) - zf(1)$$

$$Z\{f(k-1)\} = z^{-1}F(z)$$

$$Z\{f(k-2)\} = z^{-2}F(z)$$

Step 1:- Take the z-transform of the given eq^n

Step 2:- Substitute initial conditions and write the expression for F(z)

Step 3:- Take inverse z-transform of F(z) to get $\{f(k)\}$.

Ex 1 :- Solve the following difference equation

$f(k+2)+3f(k+1)+2f(k)=0$, $f(0)=0$, $f(1)=1$

$\rightarrow$ Take z transform of the given eqn

$Z\{f(k+2)\}+3Z\{f(k+1)\}+2Z\{f(k)\}=Z\{0\}=0$

$\therefore \left[z^2F(z)-z^2f(0)-zf(1)\right]+3\left[zF(z)-zf(0)\right]+2F(z)=0$

$\therefore z^2F(z)-z\times 1+3zF(z)+2F(z)=0$

$\therefore (z^2+3z+2)F(z)=z \qquad \therefore F(z)=\dfrac{z}{z^2+3z+2}$

$\therefore F(z)=\dfrac{z}{(z+1)(z+2)}=\dfrac{A}{z+1}+\dfrac{B}{z+2}$

$z=A(z+2)+B(z+1)$

Put $z=-1$ $\quad A=-1$

$z=-2$ $\quad B=2$

$\therefore F(z)=\dfrac{2}{z+2}-\dfrac{1}{z+1}$

$\therefore \{f(k)\}=2\{(-2)^{k-1}\}-\{(-1)^{k-1}\}$, $k\geq 0$.

(2) Obtain $\{f(k)\}$, give that

$12f(k+2)-7f(k+1)\}+f(k)=0$, $k\geq 0$, $f(0)=0$, $f(1)=3$

$\rightarrow$ $12Z\{f(k+2)\}-7Z\{f(k+1)\}+Z\{f(k)\}=0$

$\therefore 12\left[z^2F(z)-z^2f(0)-zf(1)\right]-7\left[zF(z)-zf(0)\right]+F(z)=0$

$\therefore 12\left[z^2F(z)-3z\right]-7zF(z)+F(z)=0$

$$\therefore \quad (12z^2-7z+1)F(z)=36z$$

$$\therefore \quad F(z)=\frac{36z}{12z^2-7z+1}=\frac{36z}{(4z-1)(3z-1)}$$

H.W use partial fraction

$$=36\left[\frac{z}{z-\frac{1}{3}}-\frac{z}{z-\frac{1}{4}}\right]$$

$$\therefore \quad \{f(k)\}=36\left[\left(\frac{1}{3}\right)^k-\left(\frac{1}{4}\right)^k\right], \quad k\geq 0.$$

3) Obtain $\{f(k)\}$, given that

$$f(k+1)+\frac{1}{2}f(k)=\left(\frac{1}{2}\right)^k, \quad k\geq 0, \quad f(0)=0$$

$$\rightarrow \quad Z\{f(k+1)\}+\frac{1}{2}Z\{f(k)\}=Z\left\{\left(\frac{1}{2}\right)^k\right\}$$

$$\therefore \quad \left(zF(z)-zf(0)\right)+\frac{1}{2}F(z)=\frac{z}{z-\frac{1}{2}}$$

$$\therefore \quad \left(z+\frac{1}{2}\right)F(z)=\frac{z}{z-\frac{1}{2}}$$

$$\therefore \quad F(z)=\frac{z}{(z-\frac{1}{2})(z+\frac{1}{2})}=\frac{A}{z-\frac{1}{2}}+\frac{B}{z+\frac{1}{2}}$$

finding A & B Homework.

Taking inverse Z-transform

$$\{f(k)\}=A\,Z^{-1}\left\{\frac{1}{z-\frac{1}{2}}\right\}+B\,Z^{-1}\left\{z+\frac{1}{2}\right\}$$

$$=A\left(\frac{1}{2}\right)^{k-1}+B\left(-\left(\frac{1}{2}\right)^{k-1}\right)$$