

4 March 2024

Fourier Transform

Fourier Series :-

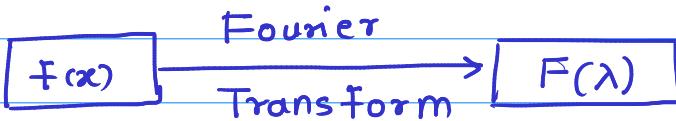
$f(x)$ is periodic fn of period $2L$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

$x \in (0, 2L) \text{ OR } (-L, L)$

$$a_0 = \frac{1}{2L} \int_0^{2L} f(x) dx, \quad a_n = \frac{1}{2L} \int_0^{2L} f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{2L} \int_0^{2L} f(x) \sin \frac{n\pi x}{L} dx$$



eg:- Time domain $\xrightarrow{\text{Trans.}}$ Frequency domain

Fourier Transform \rightarrow 1. Solve pole

$\hookrightarrow f(x)$ continuous fn 2. Solve integral equations.

Z-transform \rightarrow solve difference eqn

$\hookrightarrow \{f(k)\}$ discrete valued fn.

Defn:- Let $f(x)$ be a fn defined $(-\infty, \infty)$, then

$F\{f(x)\}$ is defined as

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-ix} dx = F(\lambda) \quad \text{--- (1)}$$

Ex:- Find FT of $f(x)$, $f(x) = e^{-5x}$, $x > 0$

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-ix} dx$$

Soln:-

$$\begin{aligned} &= \int_0^\infty e^{-5x} e^{i\lambda x} dx \\ &= \int_0^\infty e^{-(5+i\lambda)x} dx = \left[\frac{-e^{-(5+i\lambda)x}}{-(5+i\lambda)} \right]_0^\infty \end{aligned}$$

$$F(\lambda) = \frac{-e^{-\infty} + e^0}{5+i\lambda} = \frac{1}{5+i\lambda}$$

F.T
 { Exponential
 Trigonometric
 Constant
 Algebraic

Ex 2 :- Find F.T of $f(x) = \begin{cases} \sin 2x, x > 0 \\ 0, x \leq 0 \end{cases}$

$$\rightarrow F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx = \int_{-\infty}^0 0 + \int_0^{\infty}$$

$$= \int_0^{\infty} \sin 2x e^{-i\lambda x} dx$$

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) \quad a = -i\lambda, b = 2$$

$$= \left[\frac{e^{-i\lambda x}}{i^2 \lambda^2 + 4} (-i\lambda \sin 2x - 2 \cos 2x) \right]_0^\infty$$

$$= 0 - \frac{1}{4 - \lambda^2} (-2) = \frac{2}{4 - \lambda^2} \quad \begin{pmatrix} \sin 0 = 0 \\ \cos 0 = 1 \end{pmatrix}$$

Ex 3 :- Find F.T of $f(x) = \begin{cases} \cos 2x, & x > 0 \\ 0, & x < 0 \end{cases}$

$$\rightarrow F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-ix} dx$$

$$= \int_0^{\infty} \cos 2x e^{-ix} dx$$

$$a = -i\lambda$$

$$b = 2$$

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

$$= \left[\frac{-e^{i\lambda x}}{(i\lambda)^2 + 4} (-i\lambda \cos 2x + 2 \sin 2x) \right]_0^{\infty}$$

$$F(\lambda) = 0 - \frac{1}{4 - \lambda^2} (-i\lambda) = \frac{i\lambda}{4 - \lambda^2}$$

Ex 4 :- Find F.T $f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$$\rightarrow F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-ix} dx = \int_{-\infty}^0 + \int_0^{\infty}$$

$$= \int_0^{\infty} 1 \cdot e^{-ix} dx = \left[\frac{e^{-ix}}{-i} \right]_0^{\infty} = 0 + \frac{1}{i\lambda}$$

$$\therefore F\{f(x)\} = \frac{1}{i\lambda}$$

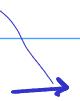
Formula :- $\int u v dx = u v_1 - u' v_2 + u'' v_3 - u''' v_4 + \dots$

dash- derivative
suffixes- integration

$u \rightarrow$ Algebraic
 $v \rightarrow$ Trigonometric or exponential

Ex :-

Find F.T of $f(x) = \begin{cases} x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$



$$F\{f(x)\} = \int_0^1 x^2 e^{-i\lambda x} dx$$

$$= \left[x^2 \left(\frac{e^{-i\lambda x}}{-i\lambda} \right) - (2x) \left(\frac{e^{-i\lambda x}}{(-i\lambda)^2} \right) + (2) \left(\frac{e^{-i\lambda x}}{(-i\lambda)^3} \right) \right]_0^1$$

$$\int e^{ax} dx = \frac{e^{ax}}{a}$$

$$\int \int e^{ax} dx dx = \frac{e^{ax}}{a^2}$$

$$\int \int e^{ax} dx dx = \frac{e^{ax}}{a^2}$$

$$\stackrel{\bullet}{l} = -1, \quad \stackrel{\bullet}{l}^2 = \stackrel{\bullet}{l} \cdot \stackrel{\bullet}{l} = -1$$

$$= \left[\left(\frac{\bar{e}^{\stackrel{\circ}{l}\lambda}}{-i\lambda} - 2 \frac{\bar{e}^{\stackrel{\circ}{l}\lambda}}{-\lambda^2} + 2 \frac{\bar{e}^{\stackrel{\circ}{l}\lambda}}{-i\lambda^3} \right) - \left(\frac{2}{-i\lambda^3} \right) \right]$$

$$= \cancel{e^{\stackrel{\circ}{l}\lambda}} \left[\frac{1}{-i\lambda} + \frac{2}{\lambda^2} - \frac{2}{i\lambda^3} \right] + \frac{2}{i\lambda^3}$$

Defn 2 :- Inverse Fourier Transform :-

I.F.T of $F(\lambda)$ is defined as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} d\lambda, \quad -\infty < \lambda < \infty \quad \text{--- (2)}$$

This eqn (2) is also called as Fourier integral representation of $f(x)$.

Ex :- Find Fourier integral representation of $f(x) = \begin{cases} 1, & 0 \leq x < \infty \\ 0, & x < 0 \end{cases}$



$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} d\lambda \quad \text{--- (1)}$$

Let us find $F(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx$

$$= \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx = \left[\frac{e^{-i\lambda x}}{-i\lambda} \right]_0^{\infty}$$

$F(\lambda) = \frac{1}{i\lambda}$

∴ The Fourier integral representation of $f(x)$ is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\lambda} e^{i\lambda x} d\lambda$$

which is required answer.

Defn3:- Fourier Cosine transform (FCT)

FCT of $f(x)$ defined for $0 < x < \infty$, as

$$F_c(\lambda) = \int_0^{\infty} f(x) \cos \lambda x dx \quad \text{--- (3)}$$

Note:- If $f(x)$ is an even function then we can find FCT

Even function: $f(-x) = f(x)$ for all x

ex: $x^2, \cos x$ etc

Ex :- Find F.C.T of $f(x) = e^{-5x}$, $x > 0$

$\rightarrow F_C(\lambda) = \int_0^{\infty} f(x) \cos \lambda x dx$

$$= \int_0^{\infty} e^{-5x} \cos \lambda x dx \quad a = -5, b = \lambda$$
$$= \left[\frac{e^{-5x}}{25 + \lambda^2} (-5 \cos \lambda x + \lambda \sin \lambda x) \right]_0^{\infty}$$
$$= 0 - \frac{1}{25 + \lambda^2} (-5) = \frac{5}{25 + \lambda^2}$$

$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$

Ex :- Find F.C.T of $f(x) = \begin{cases} x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

$\rightarrow F_C(\lambda) = \int_0^{\infty} f(x) \cos \lambda x dx$

$$= \int_0^1 x^2 \cos \lambda x dx$$
$$= \left[x^2 \left(\frac{\sin \lambda x}{\lambda} \right) - (2x) \left(-\frac{\cos \lambda x}{\lambda^2} \right) + (2) \left(-\frac{\sin \lambda x}{\lambda^3} \right) \right]_0^1$$
$$= \cancel{\left(\frac{\sin \lambda}{\lambda} + \frac{2 \cos \lambda}{\lambda^2} - \frac{2 \sin \lambda}{\lambda^3} \right)} - 0$$
$$F_C(\lambda) = \frac{\sin \lambda}{\lambda} + \frac{2 \cos \lambda}{\lambda^2} - \frac{2 \sin \lambda}{\lambda^3}$$

Defn 4:- Fourier Sine transform (FST), $0 < x < \infty$

$$F_s(\lambda) = \int_0^{\infty} f(x) \sin \lambda x \, dx \quad \text{--- (1)}$$

Note:- If $f(x)$ is an odd fn ie $f(-x) = -f(x)$ then we

find $F_s(\lambda)$

Ex:- Find the FST of $f(x) = e^{-4x}$, $x > 0$

$$\rightarrow F_s(\lambda) = \int_0^{\infty} f(x) \sin \lambda x \, dx$$

$$= \int_0^{\infty} e^{-4x} \sin \lambda x \, dx$$
$$= \left[\frac{-e^{-4x}}{16 + \lambda^2} (-4 \sin \lambda x - \lambda \cos \lambda x) \right]_0^{\infty}$$

$$= 0 - \frac{1}{16 + \lambda^2} (-\lambda) = \frac{\lambda}{16 + \lambda^2}$$

Ex:- Find FST of $f(x) = \begin{cases} x^2, & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$

$$\rightarrow F_s(\lambda) = \int_0^{\infty} f(x) \sin \lambda x \, dx$$

$$= \int_0^1 x^2 \sin \lambda x \, dx$$

$$= \left[x^2 \left(-\frac{\cos \lambda x}{\lambda} \right) - (2x) \left(-\frac{\sin \lambda x}{\lambda^2} \right) + (2) \left(\frac{\cos \lambda x}{\lambda^3} \right) \right]_0^1$$

$$= \left(-\frac{\cos \lambda}{\lambda} + \frac{2 \sin \lambda}{\lambda^2} + \frac{2 \cos \lambda}{\lambda^3} \right) - \left(\frac{2}{\lambda^3} \right)$$