Ex :- Find the Fourier sine and cosine transforms of the following functions

1) $F(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$

Sol^n:-

$$F_s(\lambda) = \int_0^\infty F(x) \sin \lambda x \, dx = \int_0^1 1 \cdot \sin \lambda x \, dx + \int_1^\infty 0 \, dx$$

$$= \left[\frac{-\cos \lambda x}{\lambda} \right]_0^1 = \frac{-\cos \lambda}{\lambda} + \frac{1}{\lambda} = \frac{1-\cos \lambda}{\lambda}$$

$$F_c(\lambda) = \int_0^\infty F(x) \cos \lambda x \, dx$$

$$= \int_0^1 1 \cdot \cos \lambda x \, dx + \int_1^\infty 0 \, dx$$

$$= \left[\frac{\sin \lambda x}{\lambda} \right]_0^1 = \frac{\sin \lambda}{\lambda}$$

Ex 2:- Find the Fourier sine and cosine transform of

$f(x) = 2 e^{-5x} + 5 e^{-2x}, \quad x > 0$

Sol^n:-

$$F_s(\lambda) = \int_0^\infty F(x) \sin \lambda x \, dx$$

$$= \int_0^\infty \left(2 e^{-5x} + 5 e^{-2x}\right) \sin \lambda x \, dx$$

$$= 2 \int_0^\infty e^{-5x} \sin \lambda x \, dx + 5 \int_0^\infty e^{-2x} \sin \lambda x \, dx$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin \lambda x - b \cos \lambda x)$$

$$= 2\left[\frac{e^{-5x}}{25+\lambda^2}(-5\sin\lambda x - \lambda\cos\lambda x)\right]_0^\infty + 5\left[\frac{e^{-2x}}{4+\lambda^2}(-2\sin\lambda x - \lambda\cos\lambda x)\right]_0^\infty$$

$$= 2\left[0 - \frac{1}{25+\lambda^2}(-\lambda)\right] + 5\left[0 - \frac{(-\lambda)}{4+\lambda^2}\right]$$

$$F_s(\lambda) = \frac{2\lambda}{25+\lambda^2} + \frac{5\lambda}{4+\lambda^2}$$

$$F_c(\lambda) = \int_0^\infty f(x)\cos\lambda x\,dx$$

$$= \int_0^\infty (2e^{-5x} + 5e^{-2x})\cos\lambda x\,dx$$

$$= 2\int_0^\infty e^{-5x}\cos\lambda x\,dx + 5\int_0^\infty e^{-2x}\cos\lambda x\,dx$$

$$= 2\left[\frac{e^{-5x}}{25+\lambda^2}(-5\cos\lambda x + \lambda\sin\lambda x)\right]_0^\infty + 5\left[\frac{e^{-2x}}{4+\lambda^2}(-2\cos\lambda x + \lambda\sin\lambda x)\right]_0^\infty$$

$$= 2\left[0 + \frac{5}{25+\lambda^2}\right] + 5\left[0 + \frac{2}{4+\lambda^2}\right]$$

$$F_c(\lambda) = 10\left[\frac{1}{25+\lambda^2} + \frac{1}{4+\lambda^2}\right]$$

Def^{n}5 Inverse Fourier Cosine transform :- (I.F.C.T)

I.F.C.T of $F_c(\lambda)$ is defined as

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\,d\lambda \quad \text{———} \quad (5)$$

This eq^n (5) also called Fourier cosine integral representation of $f(x)$.

Def^n 6:- Inverse Fourier sine transform (IFST)

I.F.S.T of $F_s(\lambda)$ is defined as

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\, d\lambda \quad \text{—— (6)}$$

This eq^n (6) is also called as Fourier sine integral representation of $f(x)$.

Ex:- Solve the following integral equation

$$\int_0^\infty f(x)\cos\lambda x\, dx = e^{-\lambda},\ \lambda > 0$$

Sol^n:- $F_c(\lambda) = \int_0^\infty f(x)\cos\lambda x\, dx$

Given : $F_c(\lambda) = e^{-\lambda},\ \lambda > 0$

By using I.F.C.T we have

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty e^{-\lambda}\cos\lambda x\, d\lambda \qquad a=-1,\ b=x$$

$$= \frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}(-\cos\lambda x + x\sin\lambda x)\right]_0^\infty$$

$$= \frac{2}{\pi}\left[0 - \frac{1}{1+x^2}(-1+0)\right]$$

$$f(x) = \frac{2}{\pi}\frac{1}{1+x^2}$$

Ex 2 :- Solve the following integral eqⁿ

$$\int_0^{\infty} F(x)\sin\lambda x\,dx = e^{-\lambda}\ ,\ \lambda > 0$$

Solⁿ :- $F_s(\lambda) = e^{-\lambda}$ — given

By using I.F.S.T we have

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\lambda)\sin\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} e^{-\lambda}\sin\lambda x\,d\lambda$$

$a = -1,\ b = x$

$$= \frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}\left(-1\sin\lambda x - x\cos\lambda x\right)\right]_0^{\infty}$$

$$= \frac{2}{\pi}\left[0 - \frac{1}{1+x^2}(0 - x)\right]$$

$$f(x) = \frac{2}{\pi}\left(\frac{x}{1+x^2}\right)$$

Ex 3 :- Solve the following integral eqⁿ

$$\int_0^{\infty} f(x)\sin\lambda x\,dx = \begin{cases} 1, & 0 \le \lambda < 1 \\ 2, & 1 \le \lambda < 2 \\ 0, & \lambda \ge 2 \end{cases}$$

→ $$F_s(\lambda) = \begin{cases} 1, & 0 \le \lambda < 1 \\ 2, & 1 \le \lambda < 2 \\ 0, & \lambda \geq 2 \end{cases}$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\lambda)\sin\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\left[\int_0^{1} 1\cdot\sin\lambda x\,d\lambda + \int_1^{2} 2\cdot\sin\lambda x\,d\lambda\right]$$

$$= \frac{2}{\pi}\left[\left(-\frac{\cos\lambda x}{x}\right)_0^1 + \left(\frac{-2\cos\lambda x}{x}\right)_1^2 \right]$$

$$= \frac{2}{\pi}\left[\left(-\frac{\cos x}{x} + \frac{1}{x}\right) + \left(-\frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right)\right]$$

$$f(x) = \frac{2}{\pi}\left[\frac{1 - 2\cos 2x + \cos x}{x}\right]$$

Ex 4:- Solve the following integral equation

$$\int_0^\infty F(x)\sin\lambda x \, dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases}$$

$\rightarrow$ $$F_s(\lambda) = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases}$$

Using I.F.S.T

$$F(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x \, d\lambda$$

$$= \frac{2}{\pi}\int_0^1 \underbrace{(1-\lambda)}_{u}\underbrace{\sin\lambda x}_{v} \, d\lambda$$

$$= \frac{2}{\pi}\left[(1-\lambda)\left(\frac{-\cos\lambda x}{x}\right) - (-1)\left(\frac{-\sin\lambda x}{x^2}\right)\right]_0^1$$

$$f(x) = \frac{2}{\pi}\left[\left(0 - \frac{\sin x}{x^2}\right) - \left(\frac{-1}{x}\right)\right] = \frac{2}{\pi}\left[\frac{x - \sin x}{x^2}\right]$$

Ex. 5 Solve

$$\int_0^\infty f(x)\cos\lambda x \, dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda > 1 \end{cases}$$

$\rightarrow$ Using I.F.C.T we have

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(\lambda)\cos\lambda x \, d\lambda$$

$$= \frac{2}{\pi}\int_0^{1}(1-\lambda)\cos\lambda x \, d\lambda$$

$$= \frac{2}{\pi}\left[(1-\lambda)\left(\frac{\sin\lambda x}{x}\right) - (-1)\left(\frac{-\cos\lambda x}{x^2}\right)\right]_0^1$$

$$= \frac{2}{\pi}\left[\left(0 - \frac{\cos x}{x^2}\right) - \left(0 - \frac{1}{x^2}\right)\right]$$

$$F(x) = \frac{2}{\pi}\left[\frac{1-\cos x}{x^2}\right]$$

Ex:- Find the Fourier sine and cosine transform of the

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0, & x > 2 \end{cases}$$

$\rightarrow$

$$F_s(\lambda) = \int_0^{\infty} f(x)\sin\lambda x \, dx$$

$$= \int_0^{1} x\sin\lambda x \, dx + \int_1^{2}(2-x)\sin\lambda x \, dx$$

$$= \left[x\left(-\frac{\cos\lambda x}{\lambda}\right) - (1)\left(-\frac{\sin\lambda x}{\lambda^2}\right)\right]_0^1 + \left[(2-x)\left(-\frac{\cos\lambda x}{\lambda}\right) - (-1)\left(-\frac{\sin\lambda x}{\lambda^2}\right)\right]_1^2$$

$$= \left[\left(-\frac{\cos\lambda}{\lambda} + \frac{\sin\lambda}{\lambda^2}\right) - (0)\right] + \left[-\frac{\sin 2\lambda}{\lambda^2} - \left(-\frac{\cos\lambda}{\lambda} - \frac{\sin\lambda}{\lambda^2}\right)\right]$$

$$= \frac{-\cos\lambda}{\lambda} + \frac{\sin\lambda}{\lambda^2} - \frac{\sin 2\lambda}{\lambda^2} + \frac{\cos\lambda}{\lambda} + \frac{\sin\lambda}{\lambda^2}$$

$$F_s(\lambda) = \frac{2\sin\lambda - \sin 2\lambda}{\lambda^2}$$

$$F_c(\lambda) = \int_0^\infty f(x)\cos\lambda x\, dx$$

$$= \int_0^1 x\cos\lambda x\, dx + \int_1^2 (2-x)\cos\lambda x\, dx$$

$$= \left[x\left(\frac{\sin\lambda x}{\lambda}\right) - (1)\left(\frac{-\cos\lambda x}{\lambda^2}\right)\right]_0^1 + \left[(2-x)\left(\frac{\sin\lambda x}{\lambda}\right) - (-1)\left(-\frac{\cos\lambda x}{\lambda^2}\right)\right]_1^2$$

$$= \left(\frac{\sin\lambda}{\lambda} + \frac{\cos\lambda}{\lambda^2}\right) - \left(0 + \frac{1}{\lambda^2}\right) + \left(0 - \frac{\cos 2\lambda}{\lambda^2}\right) - \left(\frac{\sin\lambda}{\lambda} - \frac{\cos\lambda}{\lambda^2}\right)$$

$$F_c(\lambda) = \frac{2\cos\lambda - \cos 2\lambda - 1}{\lambda^2}$$

Ex:- Find the Fourier sine and cosine transform of f^n $f(x) = e^{-x}$ and hence show that

$$\int_0^\infty \frac{\cos mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m} \quad \text{and} \quad \int_0^\infty \frac{x\sin mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m}$$

Sol^n:- $F_c(\lambda) = \int_0^\infty f(x)\cos\lambda x\, dx$

$$= \int_0^\infty e^{-x}\cos\lambda x\, dx$$

$$= \left[\frac{e^{-x}}{1+\lambda^2}\left(-\cos\lambda x + \lambda\sin\lambda x\right)\right]_0^\infty$$

$$= 0 - \frac{1}{1+\lambda^2}(-1) = \frac{1}{1+\lambda^2}$$

$$F_s(\lambda) = \int_0^\infty f(x)\sin\lambda x\,dx = \int_0^\infty e^{-x}\sin\lambda x\,dx$$

$$= \left[\frac{e^{-x}}{1+\lambda^2}(-\sin\lambda x - \lambda\cos\lambda x)\right]_0^\infty$$

$$= 0 - \frac{1}{1+\lambda^2}(-\lambda)$$

$$F_s(\lambda) = \frac{\lambda}{1+\lambda^2}$$

By using I.F.C.T,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\,d\lambda$$

$$\therefore \quad e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda^2}\cos\lambda x\,d\lambda$$

Put $x = m$, $\displaystyle\int_0^\infty \frac{\cos m\lambda}{1+\lambda^2}\,d\lambda = \frac{\pi}{2}e^{-m}$

Replace λ by x $\displaystyle\int_0^\infty \frac{\cos mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}$.

Similarly, I.F.S.T

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\,d\lambda$$

$$\frac{\pi}{2}e^{-x} = \int_0^\infty \frac{\lambda}{1+\lambda^2}\sin\lambda x\,d\lambda \qquad \text{Put } x = m$$

$$\therefore \int_0^\infty \frac{\lambda}{1+\lambda^2}\sin m\lambda\,d\lambda = e^{-m}\times\frac{\pi}{2}$$

Replace λ by x, $\displaystyle\int_0^\infty \frac{x}{1+x^2}\sin mx\,dx = \frac{\pi}{2}e^{-m}$ & hence the proof.

Ex 8 Page No: 3.20

Find the Fourier transform of

$$f(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

and hence evaluate $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\frac{x}{2}\, dx$

(Dec 2004, 2006, 2010, 2012, 2016
May 2008, 2012, ...)

→ Hint :- since $f(x)$ is an even fn

Find F.C.T

$$F_c(\lambda) = \int_0^\infty f(x)\cos\lambda x\, dx = \int_0^1 (1-x^2)\cos\lambda x\, dx$$