Unit 6 :- Numerical Methods

1) Interpolation
   1) Lagrange's interpolation
   2) Newton's interpolation

2) Numerical differentiation

3) Numerical Integration
   1) Trapezoidal rule
   2) Simpson's $\frac{1}{3}$rd rule
   3) Simpson's $\frac{3}{8}$th rule

4) Solution of ordinary diff eq^ns
   1) Euler's Method
   2) Euler's Modified Method
   3) Runge-Kutta method of order 4.

Recall :-

Algebraic & transcendental eq^ns

Simultaneous linear eq^ns.

x	x_0	x_1	x_2	x_3	$\cdots$	x_n
y	y_0	y_1	y_2	y_3	$\cdots$	y_n

Approximate $y \simeq$ polynomial function.

Value of y for some $x \in (x_0, x_n) \to$ Process of finding value of y is called interpolation.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Finite Differences :-

$x_0, x_1, x_2, \ldots x_n$ equally spaced.

$x_1 - x_0 = h, \quad x_2 - x_1 = h, \quad \ldots \quad x_n - x_{n-1} = h$

$x_0, \quad x_0 + h, \quad x_0 + 2h, \quad \ldots \quad x_0 + nh$ OR

$x_0, \quad \underset{x_1}{\underset{\shortparallel}{x_0 + h}}, \quad \underset{x_2}{\underset{\shortparallel}{x_1 + h}}, \quad \underset{x_3}{\underset{\shortparallel}{x_2 + h}}, \quad \ldots \quad x_{n-1} + h$

x	x_0,	x_1,	$\ldots$		x_n
y:	y_0	y_1	$\ldots$		y_n

Forward difference (Δ)

$\Delta f(x) = f(x+h) - f(x)$ — first order forward difference

$\Delta y_0 = y_1 - y_0$

$\Delta y_1 = y_2 - y_1$ etc.

$\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$ — 2^{nd} order forward difference

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		$y_1 - y_0 = \Delta y_0$			
x_1	y_1		$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$		
		$y_2 - y_1 = \Delta y_1$		$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$	
x_2	y_2		$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$		$\Delta^3 y_1 - \Delta^3 y_0 = \Delta^4 y_0$
		$y_3 - y_2 = \Delta y_2$		$\Delta^2 y_2 - \Delta^2 y_1 = \Delta^3 y_1$	
x_3	y_3		$\Delta y_3 - \Delta y_2 = \Delta^2 y_2$		
		$y_4 - y_3 = \Delta y_3$			
x_4	y_4				

Ex:- Write forward difference table if

x	10	20	30	40
y	1.1	2.0	4.4	7.9

Solⁿ:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
10	1.1 (y_0)			
		0.9 (Δy_0)		
20	2.0		1.5 ($\Delta^2 y_0$)	
		2.4		−0.4 ($\Delta^3 y_0$)
30	4.4		1.1	
		3.5		
40	7.9			

2) Backward difference (∇)

$$\nabla f(x) = f(x) - f(x-h)$$

$\nabla y_1 = y_1 - y_0$

$\nabla y_2 = y_2 - y_1$, - - - . .

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		$y_1 - y_0 = \nabla y_1$			
x_1	y_1		$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$		
		$y_2 - y_1 = \nabla y_2$		$\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$	
x_2	y_2		$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$		$\nabla^3 y_4 - \nabla^3 y_3 = \nabla^4 y_4$
		$y_3 - y_2 = \nabla y_3$		$\nabla^2 y_4 - \nabla^2 y_3 = \nabla^3 y_4$	
x_3	y_3		$\nabla y_4 - \nabla y_3 = \nabla^2 y_4$		
		$y_4 - y_3 = \nabla y_4$			
x_4	y_4				

Ex:- Construct the backward difference table if

	x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	10	1.1			
			0.9		
x_1	20	2.0		1.5	
			2.4		−0.4 $= \nabla^3 y_3$
x_2	30	4.4		1.1 ($\nabla^2 y_3$)	
			3.5 (∇y_3)		
x_3	40	7.9 (y_3)			

H.W. Central difference (δ)

Lagrange's interpolation :-

[If sometimes data is not equally spaced]

x	x_0	x_1	x_2	$\cdots$	x_n
y	y_0	y_1	y_2	$\cdots$	y_n

$$y \simeq f(x) = \text{polynomial } f^n \quad (\text{Lagrange's polynomial})$$

$\rightarrow$ $$y \simeq L_0(x)\, y_0 + L_1(x)\, y_1 + L_2(x)\, y_2 + \cdots + L_n(x)\, y_n$$

where $$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\cdots(x_0-x_n)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)}$$

$\vdots$

$$L_n(x) = \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})}$$

$$y \simeq L_0(x)\, y_0 + \underbrace{L_1(x)\, y_1 + \cdots + L_n(x)\, y_n}$$

$y(x_0) = y_0 \longrightarrow L_0(x_0) = 1$

$L_1(x_0) = 0, \cdots L_n(x_0) = 0$

Ex1. $\rightarrow$ Use Lagrange's interpolation formula, to find the value of y when $x = 10$, if the following values of x & y are given

$\longrightarrow$

x	$5 = x_0$	$6 = x_1$	$9 = x_2$	$11 = x_3$
y	$12 = y_0$	$13 = y_1$	$14 = y_2$	$16 = y_3$

$$y = L_0(x)\, y_0 + L_1(x)\, y_1 + L_2(x)\, y_2 + L_3(x)\, y_3$$

$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

Put $x = 10$

$$y(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$= \frac{(4)(1)(-1)}{(-1)(-4)(-6)} \times 12 + \frac{(5)(1)(-1)}{(1)(-3)(-5)} \times 13 + \frac{(5)(4)(-1)}{(4)(3)(-2)} \times 14 + \frac{(5)(4)(1)}{(6)(5)(2)} \times 16$$

$$= 14.66$$

2) The following table gives the viscosity of an oil as a function of temperature. Use Lagrange's formula to find viscosity of oil at a temperature of 140°.

→

Temp° (t):	$110 = t_0$	$130 = t_1$	$160 = t_2$	$190 = t_3$
Viscosity (V):	$10.8 = V_0$	$8.1 = V_1$	$5.5 = V_2$	$4.8 = V_3$

$$V(t) = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)} V_0 + \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)} V_1$$

$$+ \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)} V_2 + \frac{(t-t_0)(t-t_1)(t-t_2)}{(t_3-t_0)(t_3-t_1)(t_3-t_2)} V_3$$

Put $t = 140$

$$V(140) = \frac{(140-130)(140-160)(140-190)}{(110-130)(110-160)(110-190)} \times 10.8 + \frac{(140-110)(140-160)(140-190)}{(130-110)(130-160)(130-190)} 8.1$$

$$+ \frac{(140-110)(140-130)(140-190)}{(160-110)(160-130)(160-190)} \times 5.5 + \frac{(140-110)(140-130)(140-160)}{(190-110)(190-130)(190-160)} \times 4.8$$

$V(140) = 7.03$

Ex3:- Find the polynomial $f(x)$ by using Lagrange's Formula and hence find $f(3)$ for

x	$0 = x_0$	$1 = x_1$	$2 = x_2$	$5 = x_3$
$f(x)$	2	3	12	147

$$\rightarrow \quad f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} \times 2 + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} \times 3$$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} \times 12 + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} \times 147$$

$$f(x) = x^3 + x^2 - x + 2 \quad \longrightarrow \text{Check?}$$

$$f(3) = 27 + 9 - 3 + 2 = 35$$

Newton's Forward interpolation Formula :-

1) Forward difference operator (Δ)

$$\Delta f(x) = f(x+h) - f(x)$$

2) Backward difference operator (∇)

$$\nabla f(x) = f(x) - f(x-h)$$

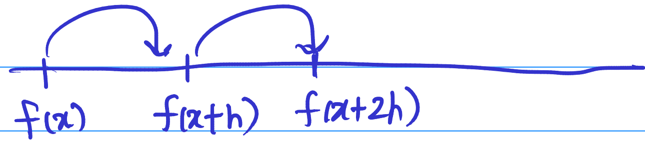
3) Shifting operator (E)

Defⁿ:- $E f(x) = f(x+h)$

$$E^{2} f(x) = E(E f(x)) = E(f(x+h)) = f(x+2h)$$

$$\vdots$$

$$E^{n} f(x) = f(x+nh)$$



$$E^{-1} f(x) = f(x-h)$$

$$\vdots$$

$$E^{-n} f(x) = f(x-nh)$$

4) **Relation of Δ and E**

$$\Delta f(x) = f(x+h) - f(x)$$

$$= E f(x) - f(x) = (E-1) f(x)$$

$$\therefore \boxed{\Delta = E - 1 \quad \text{OR} \quad E = 1 + \Delta}$$

5) **Relation of ∇ and E**

$$\nabla f(x) = f(x) - f(x-h) = f(x) - E^{-1} f(x) = (1 - E^{-1}) f(x)$$

$$\boxed{\therefore \nabla = 1 - E^{-1} \quad \text{OR} \quad E^{-1} = 1 - \nabla}$$

★ **Newton's Forward interpolation Formula :-**

x : $x_0 \quad x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_n$ ($\longrightarrow$ equally spaced)

y : $y_0 \quad y_1 \quad y_2 \quad y_3 \quad \cdots \quad y_n$

$\longrightarrow$ $x_i - x_{i-1} = h$.

Find value of y at some x.

$x = x_0 + uh \qquad \therefore u = \dfrac{x - x_0}{h}$.

$$f(x) \simeq f(x_0 + uh)$$

$$= E^{u} f(x_0)$$

$$= (1+\Delta)^{u} f(x_0) = (1+\Delta)^{u} y_0$$

$$= \left[1 + u\Delta + \frac{u(u-1)}{2!}\Delta^2 + \frac{u(u-1)(u-2)}{3!}\Delta^3 + \cdots \right] y_0$$

$$\therefore f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

Ex:- Using Newton's Forward Formula, Find the value of $f(1\cdot 6)$ if

x	1	1·4	1·8	2·2
$f(x)$	3·49	4·82	5·96	6·5

→

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	3·49 y_0			
		1·33 Δy_0		
1·4	4·82		−0·19 $\Delta^2 y_0$	
				−0·41 $\Delta^3 y_0$
1·8	5·96	1·14	−0·6	
2·2	6·5	0·54		

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$u = \frac{x - x_0}{h} = \frac{1\cdot 6 - 1}{0\cdot 4} = 1\cdot 5$$

$$= 3\cdot 49 + (1\cdot 5)(1\cdot 33) + \frac{1\cdot 5(1\cdot 5 - 1)}{2}(-0\cdot 19) + \frac{1\cdot 5(1\cdot 5-1)(1\cdot 5-2)}{6} \times -0\cdot 41$$

$$= 5\cdot 54$$

EX2: Construct Newton's forward interpolation polynomial for the following data :

x	4	6	8	10
y	1	3	8	16

Hence evaluate y for $x = 5$.

→

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1 $= y_0$			
		2 Δy_0		
6	3		3 $\Delta^2 y_0$	
		5		0

8 8 8 3

10 16

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots$$

$$u = \frac{x - x_0}{h} = \frac{x-4}{2}.$$

$$\therefore y = 1 + \left(\frac{x-4}{2}\right)\times 2 + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2} - 1\right)}{2} \times 3.$$

$$= 1 + (x-4) + \left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)\times\frac{3}{2}$$

$$= 1 + (x-4) + \frac{3}{8}(x^2 - 10x + 24)$$

$$= \frac{3}{8}x^2 - \frac{30}{8}x + 9 + x - 4 + 1 = \frac{3}{8}x^2 - \frac{22}{8}x + 6$$

$y(5) = 1.625$ (Check !)

Newton's Backward interpolation Formula :-

$$x = x_n + uh \quad \therefore \quad u = \frac{x - x_n}{h}.$$

$$\boxed{\therefore f(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots}$$

Ex:- Find $f(22)$ from the following data using Newton's backward Formulae.

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	231	204

$\longrightarrow$ $$u = \frac{x - x_n}{h} = \frac{22 - 45}{5} = -4.6$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
20	354					
25	332	-22				
30	291	-41	-19			
35	260	-31	10	29		
40	231	-29	2	-8	37	
45	204 y_n	-27 ∇y_n	2 $\nabla^2 y_n$	0 $\nabla^3 y_n$	8 $\nabla^4 y_n$	-29 $\nabla^5 y_n$

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n$$

$$+ \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 y_n$$

$$= 204 + (-4\cdot6)(-27) + \frac{(-4\cdot6)(-4\cdot6+1)}{2}\times 2 + \frac{(-4\cdot6)(-4\cdot6+1)(-4\cdot6+2)(-4\cdot6+3)}{4\times3\times2\times1}\times 8$$

$$+ \frac{(-4\cdot6)(-4\cdot6+1)(-4\cdot6+2)(-4\cdot6+3)(-4\cdot6+4)}{5\times4\times3\times2\times1}\times(-29)$$

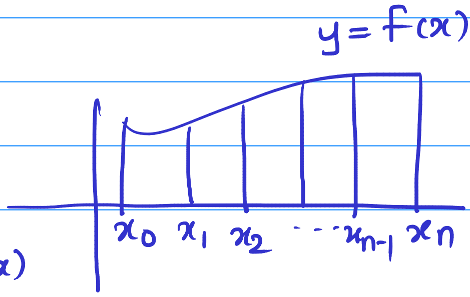
$$= \underline{352} \quad \text{(Check ?)}$$

Numerical Integration :-

Trapezoidal Rule
Simpson's $\frac{1}{3}$rd rule
Simpson's $\frac{3}{8}$th rule.

x	x_0	x_1	x_2	x_3	....	x_n
y	y_0	y_1	y_2	y_3	...	y_n

$$I = \int_{x_0}^{x_n} y\,dx = \text{Area under the curve } y=f(x) \text{ betn } x_0 \text{ to } x_n$$



$$y \simeq y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$I = \int_{x_0}^{x_n} y\,dx \qquad x_n = x_0 + nh$$

$$du = \frac{1}{h}[dx] \qquad \therefore dx = h\,du$$

$$u = \frac{x - x_0}{h} \qquad y \longrightarrow \underbrace{f(u) \longrightarrow}\text{fn of } x$$

When $x = x_0 \Rightarrow u = \dfrac{x_0 - x_0}{h} = 0$

$x = x_n \Rightarrow u = \dfrac{x_n - x_0}{h} = \dfrac{\cancel{x_0} + nh - \cancel{x_0}}{h} = n.$

$$\therefore I = \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots\right] h\,du$$

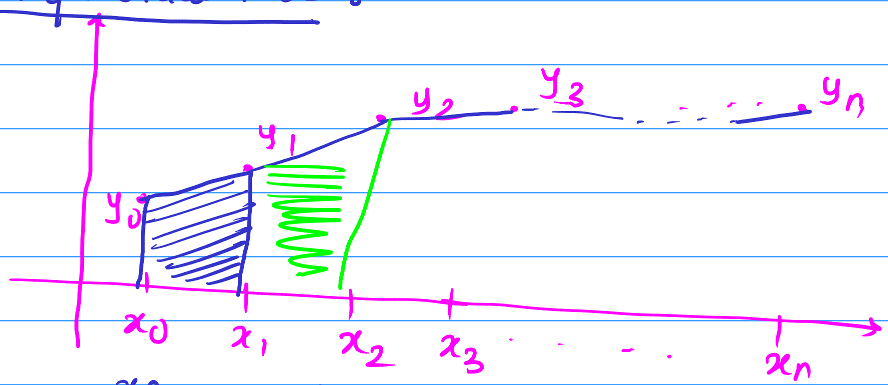
$u(u-1) = u^2 - u$

$$= h\left[u y_0 + \frac{u^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Delta^2 y_0 + \cdots\right]_0^n$$

$$I = h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \cdots\right] \quad \text{—— (1)}$$

$\hookrightarrow$ Newton's Quotes formula :-

Trapezoidal Rule :-



$n = 1$

$$I = \int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_1} y\,dx + \int_{x_1}^{x_2} y\,dx + \int_{x_2}^{x_3} y\,dx + \cdots + \int_{x_{n-1}}^{x_n} y\,dx$$

$x_0 \quad x_1$

$y_0 \quad y_1$

$y_1 - y_0 = \Delta y_0$

$$\int_{x_0}^{x_1} y\,dx = h\left[(1)\,y_0 + \frac{1}{2}(\Delta y_0)\right] \qquad \text{[Putting } n = 1 \text{ in eq}^n \text{ (1)]}$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = h\left[y_0 + \frac{1}{2}y_1 - \frac{1}{2}y_0\right] = h\left[\frac{y_0}{2} + \frac{y_1}{2}\right]$$

$$\int_{x_0}^{x_1} y\,dx = \frac{h}{2}\left[y_0 + y_1\right]$$

Similarly, $\int_{x_1}^{x_2} y\,dx = \frac{h}{2}\left[y_1 + y_2\right]$, $\int_{x_2}^{x_3} y\,dx = \frac{h}{2}\left[y_2 + y_3\right]$

$$\cdots \quad \int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}\left[y_{n-1} + y_n\right]$$

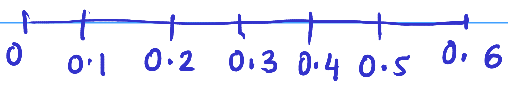
$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0 + y_1\right] + \frac{h}{2}\left[y_1 + y_2\right] + \cdots + \frac{h}{2}\left[y_{n-1} + y_n\right]$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right] \quad \text{——} \ (2)$$

This formula (2) is called Trapezoidal rule.

Ex:- Using Trapezoidal rule, find $\int_0^{0.6} e^{-x^2}\,dx$ by taking seven ordinates.

→ $y = e^{-x^2}$



x	0	0.1	0.2	0.3	0.4	0.5	0.6
y	1	0.9900	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\int_0^{0.6} e^{-x^2}\,dx = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{0.1}{2}\left[(1 + 0.6977) + 2(0.9900 + 0.9608 + 0.9139 + 0.8521 + 0.7788)\right]$$

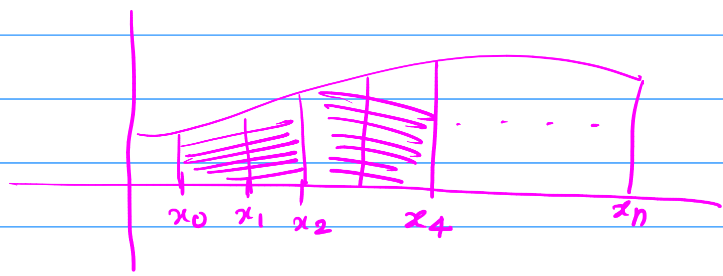
$$= 0.53445$$

Simpson's $\frac{1}{3}$rd Rule :-

$n = 2$.

Required

Even number intervals



$x_0 \quad y_0 \quad y_1 - y_0 = \Delta y_0 \quad \Delta y_1 - \Delta y_0 = \Delta^2 y_0$

$x_1 \quad y_1$

$x_2 \quad y_2 \quad y_2 - y_1 = \Delta y_1$

$$y \simeq y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0$$

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_2} y\,dx + \int_{x_2}^{x_4} y\,dx + \int_{x_4}^{x_6} y\,dx + \cdots + \int_{x_{n-2}}^{x_n} y\,dx$$

Put $n=2$ in ①

$$\int_{x_0}^{x_2} y\,dx = h\left[2y_0 + 2\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - \frac{4}{2}\right)\Delta^2 y_0\right]$$

$$\Delta y_0 = y_1 - y_0$$

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\int_{x_0}^{x_2} y\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2)$$ (Check ?)

$$\int_{x_2}^{x_4} y\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$\vdots$

$$\int_{x_{n-2}}^{x_n} y\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\right] \quad ③$$

③ is called Simpson's $\frac{1}{3}$rd rule.

Ex:- Using Simpson's $\frac{1}{3}$rd rule, evaluate $\int_0^{0.6} e^{-x^2}dx$ by taking seven ordinates.

→

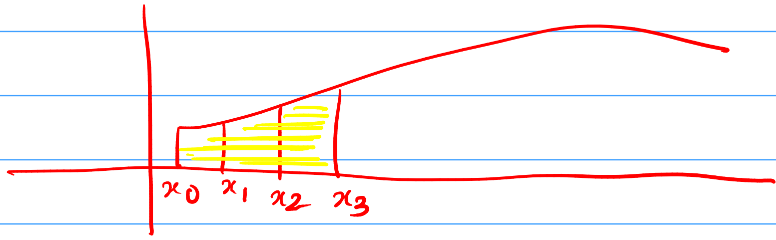
x	0	0.1	0.2	0.3	0.4	0.5	0.6
y	1	0.9900	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\int_0^{0.6} y\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{0.1}{3}\left[(1+0.6977)+4(0.9900+0.9139+0.7788)+2(0.9608+0.8521)\right]$$

$$= 0.53514.$$

Simpson's $\frac{3}{8}$th Rule



$n=3$

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_3} y\,dx + \int_{x_3}^{x_6} y\,dx + \int_{x_6}^{x_9} y\,dx + \cdots + \int_{x_{n-3}}^{x_n} y\,dx$$

$$= \frac{3h}{8}\left[(y_0+y_n)+2(y_3+y_6+y_9+\cdots+y_{n-3}) + 3(y_1+y_2+y_4+y_5+\cdots+y_{n-1})\right]$$

Ex Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using Simpson's $\frac{3}{8}$th rule.

→ $y = \frac{1}{1+x^2}$

x	0	1	2	3	4	5	6
y	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3}{8}\left[(1+0.027)+2(0.1)+3(0.5+0.2+0.0588+0.0385)\right]$$

$$= 1.3571$$

Trapezoidal rule

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots+y_{n-1})\right]$$

Simpson's $\frac{1}{3}$rd rule

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\cdots+y_{n-2})+4(y_1+y_3+\cdots+y_{n-1})\right]$$

Simpson's $\frac{3}{8}$th rule

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+\cdots+y_{n-1})+2(y_3+y_6+\cdots+y_{n-3})\right]$$

Ex :- The table below shows the temperature F(t) as a function of time :

t	1	2	3	4	5	6	7
$F(t)$	81	75	80	83	78	70	60
	f_0	f_1	f_2	f_3	f_4	f_5	f_6

Using Simpson's $\frac{1}{3}$rd rule, estimate $\int_1^7 F(t)\,dt$

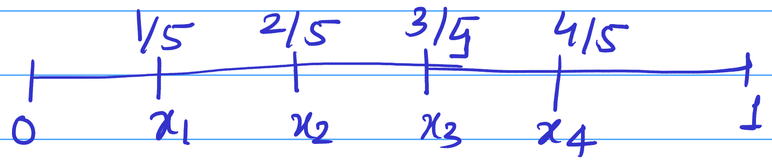
Solⁿ:-

$$\int_1^7 f(t)\,dt = \frac{h}{3}\left[(F_0+F_6)+4(F_1+F_3+F_5)+2(F_2+F_4)\right]$$

$$= \frac{1}{3}\left[(81+60)+4(75+83+70)+2(80+78)\right]$$

$$\Rightarrow 456.33.$$

Ex:- Use Trapezoidal rule to evaluate $\int_0^1 x^3\,dx$ considering five subintervals.



x	0	1/5	2/5	3/5	4/5	1
$y=x^3$	0	0.008	0.064	0.216	0.512	1
	y_0	y_1	y_2	y_3	y_4	y_5

$$I = \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{1/5}{2}\left[(0+1)+2(0.008+0.064+0.216+0.512)\right]$$

$$= \underline{0.26}$$

$\underline{\text{Exact Ans}} = \int_0^1 x^3\,dx$

$$= \left(\frac{x^4}{4}\right)_0^1 = \frac{1}{4} = \underline{0.25}$$

H.W:- Evaluate $\int_0^1 \frac{dx}{1+x}$ using

1) Trapezoidal rule
2) Simpson's $\frac{1}{3}$rd rule
3) Simpson's $\frac{3}{8}$th rule.