Tutorial No. 10.

Q1.] Solve the following.

i) Solve by guass - elimination method, the system of equations

$$2x_1 + x_2 + x_3 = 10$$
$$3x_1 + 2x_2 + 3x_3 = 18$$
$$x_1 + 4x_2 + 9x_3 = 16.$$

$$\rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 4 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$$

$$R_{13} \quad \begin{bmatrix} 1 & 4 & 9 \\ 3 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 18 \\ 10 \end{bmatrix}$$

$$\begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix} \quad \begin{bmatrix} 1 & 4 & 9 \\ 0 & -10 & -24 \\ 0 & -7 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -30 \\ -22 \end{bmatrix}$$

$$-\frac{R_2}{2} \quad \begin{bmatrix} 1 & 4 & 9 \\ 0 & 5 & 12 \\ 0 & -7 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 15 \\ -22 \end{bmatrix}$$

$$5R_3 + 7R_2 \quad \begin{bmatrix} 1 & 4 & 9 \\ 0 & 5 & 12 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 15 \\ -5 \end{bmatrix}$$

$$x + 4y + 9z = 16 \implies 7$$
$$5y + 12z = 15 \implies -9$$
$$-z = -5 \implies 5$$

2.) Use Triangular factorisation method.

$$2x_1 + 2x_2 + 3x_3 = 4$$
$$4x_1 - 2x_2 + x_3 = 9$$
$$x_1 + 5x_2 + 4x_3 = 3$$

→ Given System can be written in matrix form as
$Ax = B$

where $A = \begin{bmatrix} 2 & 2 & 3 \\ 4 & -2 & 1 \\ 1 & 5 & 4 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 9 \\ 3 \end{bmatrix}$

Now, we express $A = LU$

$$\begin{bmatrix} 2 & 2 & 3 \\ 4 & -2 & 1 \\ 1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Multiplying R.H.S & equating the elements in corresponding position, we get.

i) $u_{11} = 2$, $u_{12} = 2$, $u_{13} = 3$

ii) $l_{21}u_{11} = 4 \Rightarrow l_{21} = \frac{4}{2} = 2$; $l_{31}u_{11} = 1 \Rightarrow l_{31} = \frac{1}{2}$

iii) $l_{21}u_{12} + u_{22} = -2$ and $l_{21}u_{13} + u_{23} = 1$

$\therefore u_{22} = -2 - (2)(2) = -6$ & $u_{23} = 1 - (2)(3) = -5$

iv) $l_{31}u_{12} + l_{32}u_{22} = 5$ & $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4$

$\therefore l_{32} = \frac{-2}{3}$ & $u_{33} = -\frac{5}{6}$

Thus, we have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1/2 & -2/3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 2 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5/6 \end{bmatrix}$$

Given system can be written as

$$LUX = B$$

Let $UX = Z$ $\therefore LZ = B$ where $Z = [z_1, z_2, z_3]^T$

$LZ = B$ gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1/2 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 3 \end{bmatrix}$$

which is equivalent to

$$z_1 = 4, \quad 2z_1 + z_2 = 9, \quad \frac{1}{2} z_1 - \frac{2}{3} z_2 + z_3 = 3$$

using forward substitution

$$z_1 = 4, \quad z_2 = 1 \ \& \ z_3 = 5/3.$$

Hence, solution of original system

$UX = Z$ as

$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5/6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5/3 \end{bmatrix}$$

$$2x_1 + 2x_2 + 3x_3 = 4, \quad -6x_2 - 5x_3 = 1, \quad -5/6\, x_3 = 5/3$$

$$\Rightarrow x_1 = -\frac{7}{2}, \quad x_2 = \frac{3}{2}, \quad x_3 = -2$$

3.) Solve following using cholesky's method

$4x_1 + 2x_2 + 14x_3 = 14$

$2x_1 + 17x_2 - 5x_3 = -101$

$14x_1 - 5x_2 + 83x_3 = 155$

$Ax = B$

→ $A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$

let $A = LL^T$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$l_{11}^2 = 4$ $l_{11} = 2$

$l_{11} l_{21} = 2$ $l_{21} = 1$

$l_{11} l_{31} = 14$ $l_{31} = 7$

$l_{21}^2 + l_{22}^2 = 17$ $l_{22} = 4$

$l_{21} l_{31} + l_{22} l_{32} = -5$ $l_{32} = -3$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 83$ $l_{33} = 5$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix}$$

$LL^T x = B$, let $L^T x = z$ $\therefore LZ = B$ $z = [z_1, z_2, z_3]^T$

$LZ = B$ → $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$

$z_1 = 7$, $z_2 = -27$ & $z_3 = 5$

let $L^T x = z$

$\therefore x_1 = 3$, $x_2 = -6$ & $x_3 = 1$

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4.) Solve by Jacobi's iteration method, the equations

$$10x + y - z = 11.19$$
$$x + 10y + z = 28.08$$
$$-x + y + 10z = 35.61$$

correct to two decimal places.

→

$$x = \frac{1}{10}(11.19 - y + z)$$

$$y = \frac{1}{10}(28.08 - x - z)$$

$$z = \frac{1}{10}(35.61 + x - y)$$

$x_0, y_0, z_0 = 0$

i=1

$$\therefore x_1 = \frac{1}{10}(11.19 - y_0 + z_0) = \frac{11.19}{10} = 1.19$$

$$y_1 = \frac{1}{10}(28.08 - x_0 - z_0) = \frac{28.08}{10} = 2.808$$

$$z_1 = \frac{1}{10}(35.61 + x_0 - y_0) = \frac{35.61}{10} = 3.561$$

i=2

$$x_2 = \frac{1}{10}(11.19 - y_1 + z_1) = 1.19$$

$$y_2 = \frac{1}{10}(28.08 - x_1 - z_1) = 2.34$$

$$z_2 = \frac{1}{10}(35.61 + x_1 - y_1) = 3.39$$

i=3

$$x_3 = \frac{1}{10}(11.19 - y_2 + z_2) = 1.22$$

$$y_3 = \frac{1}{10}(28.08 - x_2 - z_2) = 2.35$$

$$z_3 = \frac{1}{10}(35.61 + x_2 - y_2) = 3.45$$

i=4

$$x_4 = \frac{1}{10}(11.19 - y_3 + z_3) = 1.23$$

$$y_4 = \frac{1}{10}(28.08 - x_3 - z_3) = 2.34$$

$$z_4 = \frac{1}{10}(35.61 + x_3 - y_3) = 3.45$$

i=5

$x_5 = 1.23$
$y_5 = 2.34$
$z_5 = 3.45$

$\therefore$ $x = 1.23$, $y = 2.34$, $z = 3.45$

5) $20x + y - 2z = 17$
$3x + 20y - z = -18$
$2x - 3y + 20z = 25$
Solve using Gauss-seidal method.

$\rightarrow$

$$x = \frac{1}{20}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z)$$

$$z = \frac{1}{20}(25 - 2x + 3y)$$

$x_0 = 0 , \quad y_0 = 0 , \quad z_0 = 0$

i=1

$$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500$$

$$y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = -1.0275$$

$$z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0109$$

i=2

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.00025$$

$$y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$$

$$z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 0.9998$$

i=3

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$$

$$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0000$$

$$z_3 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 1.0000$$