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Assignment on z transform

Q1. Find $z\{f(k)\}$ where $f(k) = \begin{cases} 3^k, & k<0 \\ 2^k, & k \geq 0 \end{cases}$

Soln: By defn, $Z[f(k)] = \sum_{z=-\infty}^{\infty} [f(k)\, z^{-k}]$

$\therefore z[f(k)] = \sum_{z=-\infty}^{-1} [3^k z^{-k}] + \sum_{z=0}^{\infty} [2^k z^{-k}]$

For 1st term,

put $r = -k$

for $z = -\infty,\ r = \infty$

$z = -1,\ r = 1$

$\therefore z[f(k)] = \sum_{r=1}^{\infty} [3^{-r} z^{r}] + \sum_{z=0}^{\infty} [2^k z^{-k}]$

$= \sum_{r=1}^{\infty} [3^{-1} z]^r + \sum_{z=0}^{\infty} [2 z^{-1}]^k$

$\therefore$ This is an GP,

for first term,

$a = 3^{-1} z$

$r = 3^{-1} z$

for second term,

$a = 1$

$r = 2z^{-1}$

$S_\infty = \dfrac{a}{1-r}$

$\therefore z[f(k)] = \dfrac{3^{-1} z}{1-3^{-1} z} + \dfrac{1}{1-2z^{-1}}$

$= \dfrac{z}{3-z} + \dfrac{z}{z-2}$ is the z-transform

$3-z>0$	$z-2>0$
$z<3$	$z>2$
$\lvert z\rvert<\lvert 3\rvert$	$\lvert z\rvert>\lvert 2\rvert$
$\lvert z\rvert<3$	$\lvert z\rvert>2$

$\therefore 2<\lvert z\rvert<3$

Q2. Find $z[f(k)]$ where

i) $\frac{2^k}{2!}$, $k \geq 0$

Soln: By formula,

$$z\left[\frac{a^k}{k!}\right] = e^{a/2}, \quad k \geq 0$$

$$z\left[\frac{2^k}{2!}\right] = e^{2/2}$$

$= e$ is the z transform

ii) $f(k) = e^{-ak}$, $k \geq 0$

Soln: By formula,

$$z[f(k)] = \sum_{z=-\infty}^{\infty} [f(k) z^{-k}]$$

$$= \sum_{z=0}^{\infty} [e^{-ak} z^{-k}]$$

$$= \sum_{z=0}^{\infty} [e^{-a} z^{-1}]^k$$

$\therefore$ This is GP,

$a = 1$

$r = e^{-a} z^{-1}$

$$S_\infty = \frac{a}{1-r}$$

$$z[f(k)] = \frac{1}{1-e^{-a}z^{-1}}$$

$$z[f(k)] = \frac{e^a z}{e^a z - 1}$$ is the z transform

$\therefore$ $e^a z - 1 > 0$

$e^a z > 1$

$z > e^{-a}$

$|z| > |e^{-a}|$

Q3 Find $Z[f(k)]$ of

i) $f(k) = \frac{\sin ak}{k}$, $k > 0$

Soln: By formula,

$$Z[\sin ak] = \frac{z \sin a}{z^2 - 2z\cos a + 1}$$

$$Z\left[\frac{\sin ak}{k}\right] = \int_z^{\infty} \frac{1}{z} \times \frac{\sin a \times z}{z^2 - 2z\cos a + 1} \, dz$$

$$= \int_z^{\infty} \frac{\sin a}{z^2 - 2z\cos a + 1} \, dz$$

$$= \int_z^{\infty} \frac{\sin a}{z^2 - 2z\cos a + \cos^2 a + \sin^2 a} \, dz$$

$$= \int_z^{\infty} \frac{\sin a}{(z - \cos a)^2 + \sin^2 a} \, dz$$

$$= \cancel{\sin a} \left[\frac{1}{\cancel{\sin a}} \tan^{-1}\left(\frac{z - \cos a}{\sin a}\right)\right]_z^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{z - \cos a}{\sin a}\right)$$

$$= \cot^{-1}\left(\frac{z - \cos a}{\sin a}\right)$$

ii) $f(k) = \frac{2^k}{k}$, $k \geq 1$

Soln: By formula,

$$Z[f(k)] = \sum_{z=-\infty}^{\infty} [f(k) z^{-k}]$$

$$= \sum_{z=1}^{\infty} [2^k z^{-k}]$$

$$= \sum_{z=1}^{\infty} [2z^{-1}]^k$$

$\therefore$ This is GP,

$a = 2z^{-1}$

$r = 2z^{-1}$

$$S_\infty = \frac{a}{1-r}$$

$$\therefore Z[f(k)] = \frac{2z^{-1}}{1-2z^{-1}}$$

$$Z[2^k] = \frac{1}{\frac{z}{2}-1}$$

$$\therefore \frac{z}{2} - 1 > 0$$

$$z > 2$$

$$|z| > |2|$$

$$|z| > 2$$

$$Z\left[\frac{2^k}{k}\right] = \int_z^\infty \frac{1}{z} \times \frac{2}{z-2}\, dz$$

$$\cancel{= 2\int_z^\infty \frac{1}{z(z-2)}\, dz}$$

$$\cancel{= 2\int_z^\infty \left(\frac{1/2}{z} - \frac{1/2}{z+2}\right) dz}$$

$$\cancel{= 2\left[\frac{1}{z(z+2)} - \frac{1}{2z}\right]}$$

$$\cancel{= \left[\log(z) - \log(z+2)\right]_z^\infty}$$

$$\cancel{Z\left[\frac{2^k}{k}\right] = \log(z+2) - \log z}$$

$$\cancel{= \log\left(\frac{z+2}{z}\right)}$$

$$= 2\int_z^\infty \frac{1}{z(z-2)}\, dz$$

$$= 2\int_z^\infty \left(\frac{1/2}{z} - \frac{1/2}{z-2}\right) dz$$

$$= \left[\log z - \log(z-2)\right]_z^\infty$$

$$= -\log\left(\frac{z-2}{z}\right)$$

$\cancel{z-2 \Rightarrow}$

$$= \log\left(\frac{z}{z-2}\right)$$

$$z-2>0$$

$$|z|>|2|$$

$$\therefore |z|>2$$

Q4. Find $z^{-1}\left[\frac{1}{(z-3)(z-2)}\right]$ $\quad 2<|z|<3$

Soln: multiply by z^{k-1}

$$z^{k-1}[F(z)] = \frac{z^{k-1}}{(z-2)(z-3)}$$

$\therefore$ poles are $z=3;\ z=2$

Let R_1 = residue at pole $z=2$

$$R_1 = [(z-2)z^{k-1}F(z)]_{z=2}$$

$$= \left[\frac{\cancel{(z-2)}\,z^{k-1}}{\cancel{(z-2)}(z-3)}\right]_{z=2}$$

$$= \frac{2^{k-1}}{-1}$$

$$R_1 = -2^{k-1}$$

Let R_2 = residue at pole $z=3$

$$R_2 = [(z-3)z^{k-1}F(z)]$$

$$= \left[\frac{\cancel{(z-3)}\,z^{k-1}}{(z-2)\cancel{(z-3)}}\right]_{z=3}$$

$$R_2 = 3^{k-1}$$

$\cancel{z f(k)}$ $\quad z^{-1}[F(z)] = f(k)$

$$\therefore f(k) = R_1 + R_2$$

$$f(k) = 3^{k-1} - 2^{k-1}$$

Q5. Find $z^{-1}\left(\frac{z}{z-5}\right)$ if $|z|>5$ and $|z|<5$

Solⁿ: ~~multiply by z^{k-1},~~

~~$F(z) = \frac{z \times z^{k-1}}{z-5}$~~

~~$F(z) = \frac{z^k}{z-5}$~~

1) $|z|>5$

By formula,

$$z^{-1}\left(\frac{z}{z-a}\right) = \left\{ a^k, \quad |z|>|a| \right.$$

$$\therefore z^{-1}\left(\frac{z}{z-5}\right) = 5^k$$

2) $|z|<5$

By formula,

$$z^{-1}\left(\frac{z}{z-a}\right) = -a^k, \quad |z|<|a|$$

$$z^{-1}\left(\frac{z}{z-a}\right) = -5^k$$

Q6. Find $z^{-1}\left[\frac{z}{(z-\frac{1}{4})(z-\frac{1}{5})}\right]$, $|z|>\frac{1}{4}$

Solⁿ: multiply by z^{k-1},

$$z^{k-1}[F(z)] = \frac{z^{k-1} \times z}{(z-\frac{1}{4})(z-\frac{1}{5})}$$

$$= \frac{z^k}{(z-\frac{1}{4})(z-\frac{1}{5})}$$

$\therefore$ poles are $z=\frac{1}{4}$, $z=\frac{1}{5}$

Let R_1 = residue at $z=\frac{1}{4}$

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$$R_1 = [(z-\tfrac{1}{4})\, z^{k-1} F(z)]_{z=\frac{1}{4}}$$

$$= \left[\frac{\cancel{(z-\frac{1}{4})}\, z \times z^{k-1}}{\cancel{(z-\frac{1}{4})}(z-\frac{1}{5})}\right]_{z=\frac{1}{4}}$$

$$= \frac{(\frac{1}{4})^k}{\frac{1}{4}-\frac{1}{5}}$$

$$R_1 = 20\left(\frac{1}{4}\right)^k$$

Let R_2 = residue at pole $z = \frac{1}{5}$

$$R_2 = [(z-\tfrac{1}{5})\, z^{k-1} F(z)]_{z=\frac{1}{5}}$$

$$= \left[\frac{\cancel{(z-\frac{1}{5})}\, z \times z^{k-1}}{(z-\frac{1}{4})\cancel{(z-\frac{1}{5})}}\right]_{z=\frac{1}{5}}$$

$$= \frac{(\frac{1}{5})^k}{\frac{1}{5}-\frac{1}{4}}$$

$$R_2 = -20\left(\frac{1}{5}\right)^k$$

$$Z^{-1}[F(z)] = f(k)$$

$$f(k) = R_1 + R_2$$

$$f(k) = 20\left(\frac{1}{4}\right)^k - 20\left(\frac{1}{5}\right)^k$$

Q7. Find ~~cos~~ $Z\left\{ \cos\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)\right\}$, $k \geq 0$

Sol^n:

$$Z\left\{\cos\left(\frac{k\pi}{2}+\frac{\pi}{4}\right)\right\} = Z\left\{\cos\frac{k\pi}{2}\cos\frac{\pi}{4} - \sin\frac{k\pi}{2}\sin\frac{\pi}{4}\right\}$$

$$= Z\left\{\frac{\cos(k\pi/2)}{\sqrt{2}} - \frac{\sin(k\pi/2)}{\sqrt{2}}\right\}$$

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Q5. Find $z^{-1}\left(\frac{z}{z-5}\right)$ if $|z|>5$ and $|z|<5$

Soln:

~~multiply by z^{k-1},~~

~~$F(z) = \frac{z \times z^{k-1}}{z-5}$~~

~~$F(z) = \frac{z^k}{z-5}$~~

1) $|z|>5$

By formula,

$$z^{-1}\left(\frac{z}{z-a}\right) = \{a^k\}, \quad |z|>|a|$$

$$\therefore z^{-1}\left(\frac{z}{z-5}\right) = 5^k$$

2) $|z|<5$

By formula,

$$z^{-1}\left(\frac{z}{z-a}\right) = -a^k, \quad |z|<|a|$$

$$z^{-1}\left(\frac{z}{z-a}\right) = -5^k$$

Q6. Find $z^{-1}\left[\frac{z}{(z-\frac{1}{4})(z-\frac{1}{5})}\right]$, $|z|>\frac{1}{4}$

Soln:

multiply by z^{k-1},

$$z^{k-1}[F(z)] = \frac{z^{k-1} \times z}{(z-\frac{1}{4})(z-\frac{1}{5})}$$

$$= \frac{z^k}{(z-\frac{1}{4})(z-\frac{1}{5})}$$

$\therefore$ poles are $z=\frac{1}{4}, z=\frac{1}{5}$

Let R_1 = residue at $z=\frac{1}{4}$

$$R_1 = \left[\left(z-\tfrac{1}{4}\right) z^{k-1} F(z) \right]_{z=\frac{1}{4}}$$

$$= \left[\frac{\cancel{(z-\tfrac{1}{4})}\, 2 \times z^{k-1}}{\cancel{(z-\tfrac{1}{4})}\left(z-\tfrac{1}{5}\right)} \right]_{z=\frac{1}{4}}$$

$$= \frac{\left(\frac{1}{4}\right)^k}{\frac{1}{4}-\frac{1}{5}}$$

$$R_1 = 20\left(\frac{1}{4}\right)^k$$

Let R_2 = residue at pole $z = \frac{1}{5}$

$$R_2 = \left[\left(z-\tfrac{1}{5}\right) z^{k-1} F(z) \right]_{z=\frac{1}{5}}$$

$$= \left[\frac{\cancel{(z-\tfrac{1}{5})}\, 2 \times z^{k-1}}{\left(z-\tfrac{1}{4}\right)\cancel{(z-\tfrac{1}{5})}} \right]_{z=\frac{1}{5}}$$

$$= \frac{\left(\frac{1}{5}\right)^k}{\frac{1}{5}-\frac{1}{4}}$$

$$R_2 = -20\left(\frac{1}{5}\right)^k$$

$$z^{-1}[F(z)] = f(k)$$

$$f(k) = R_1 + R_2$$

$$f(k) = 20\left(\frac{1}{4}\right)^k - 20\left(\frac{1}{5}\right)^k$$

Q7. Find ~~cos~~ $z\left\{ \cos\left(\frac{k\pi}{2} + \frac{\pi}{4}\right) \right\},\ k \geq 0$

Soln:

$$z\left\{\cos\left(\frac{k\pi}{2}+\frac{\pi}{4}\right)\right\} = z\left\{\cos\frac{k\pi}{2}\cos\frac{\pi}{4} - \sin\frac{k\pi}{2}\sin\frac{\pi}{4}\right\}$$

$$= z\left\{\frac{\cos(k\pi/2)}{\sqrt{2}} - \frac{\sin(k\pi/2)}{\sqrt{2}}\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{z(z-\cos\pi/2)}{z^2-2z\cos\pi/2+1} - \frac{z\sin\pi/2}{z^2-2z\cos\pi/2+1}\right\}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right]$$

$$Z\left\{\cos\left(\frac{k\pi}{2}+\frac{\pi}{4}\right)\right\} = \frac{z^2-z}{\sqrt{2}(z^2+1)}$$

Q8. Using inversion ~~...~~ integral method, find:

$$Z^{-1}\left(\frac{z}{(z-1)(z-2)}\right)$$

Soln:

$$F(z) = \frac{z}{(z-1)(z-2)}$$

multiply by z^{k-1},

$$z^{k-1}F(z) = \frac{z^{k-1}\times z}{(z-1)(z-2)}$$

$$= \frac{z^k}{(z-1)(z-2)}$$

$\therefore$ poles are $z=1$; $z=2$

Let R_1 = residue for pole $z=1$

$$R_1 = [(z-1)z^{k-1}F(z)]_{z=1}$$

$$= \left[\frac{\cancel{(z-1)}z^{k-1}\times z}{\cancel{(z-1)}(z-2)}\right]_{z=1}$$

$$= \frac{1^k}{-1}$$

$$R_1 = -1$$

Let R_2 = residue for pole $z=2$

$$R_2 = [(z-2)z^{k-1}F(z)]_{z=2}$$

$$= \left[\frac{\cancel{(z-2)}z^{k-1}\times z}{(z-1)\cancel{(z-2)}}\right]_{z=2}$$

$$= \frac{2^k}{1}$$

$$R_2 = 2^k$$

$$Z^{-1}[F(z)] = f(k)$$
$$\therefore f(k) = R_1 + R_2$$
$$f(k) = 2^k - 1$$

Q9. Find : $Z^{-1}\left(\frac{1}{(z-4)(z-5)}\right)$ by inversion integral method.

Soln.
$$F(z) = \frac{1}{(z-4)(z-5)}$$

multiply by z^{k-1}
$$z^{k-1}[F(z)] = \frac{z^{k-1}}{(z-4)(z-5)}$$

$\therefore$ Poles are $z=4$; $z=5$

Let R_1 = residue at pole $z=4$
$$R_1 = [(z-4)z^{k-1}F(z)]_{z=4}$$
$$= \left[\frac{\cancel{(z-4)}z^{k-1}}{\cancel{(z-4)}(z-5)}\right]_{z=4}$$
$$= \frac{4^{k-1}}{-1}$$
$$R_1 = -4^{k-1}$$

Let R_2 = residue at pole $z=5$
$$R_2 = [(z-5)z^{k-1}F(z)]_{z=5}$$
$$= \left[\frac{\cancel{(z-5)}z^{k-1}}{(z-4)\cancel{(z-5)}}\right]_{z=5}$$
$$R_2 = 5^{k-1}$$
$$Z^{-1}[F(z)] = f(k)$$
$$\therefore f(k) = R_1 + R_2$$
$$f(k) = 5^{k-1} - 4^{k-1}$$

Q10. Find Z transform of : $f(k) = (k+1)a^k$, $k \geq 0$

Soln:
$$Z[f(k)] = Z[ka^k] + Z[a^k]$$
$$= -z\left[\frac{d}{dz}\left(\frac{z}{z-a}\right)\right] + \frac{z}{z-a}$$

$$= -z\left[\frac{z-a-z}{(z-a)^2}\right] + \frac{z}{z-a}$$

$$= \frac{az}{(z-a)^2} + \frac{z}{(z-a)}$$

$$= \frac{az^2 - a^2z + z(z-a)^2}{(z-a)^3}$$

$$= \frac{az(z-a) + z(z-a)^2}{(z-a)^3}$$

$$= \frac{az + z^2 - az}{(z-a)^2}$$

$$Z[f(k)] = \frac{z^2}{(z-a)^2}$$