Recall

Unit 1:- LDE with const coeff.

$$\phi(D)y = f(x)$$

$$\phi(D) = a_0 D^n + a_1 D^{n-1} + \dots + a_n$$

$$D = \frac{d}{dx}$$

$$y = y_c + y_p$$

$$\phi(D) = 0$$

y_p

- General
- Shortcut
- Variation of parameter

$$y_p = \frac{1}{\phi(D)} f(x)$$

General Method

$$\frac{1}{D} f(x) = \int f(x)\, dx$$

$$\frac{1}{D-m} f(x) = e^{mx} \int e^{-mx} f(x)\, dx$$

$$\frac{1}{D+m} f(x) = e^{-mx} \int e^{mx} f(x)\, dx$$

Shortcut Method

$$\frac{1}{D^2+3D+2}e^{5x} = \frac{1}{(D+2)(D+1)}e^{5x}$$

$$= \frac{1}{D+2}\left[\frac{1}{D+1}e^{5x}\right]$$

$$= \frac{1}{D+2}\left[e^{-x}\int e^{x}e^{5x}\,dx\right]$$

$$= \frac{1}{D+2}\left[\frac{e^{5x}}{6}\right]$$

$$= e^{-2x}\int e^{2x}\cdot\frac{e^{5x}}{6}\,dx$$

$$= \frac{e^{5x}}{6\times 7} = \frac{e^{5x}}{42}$$

$$\frac{1}{D^2+3D+2}e^{5x} \xrightarrow[D=5]{\text{Put}} \frac{1}{42}e^{5x}$$

1) $f(x) = e^{ax}, K, a^{x}, a^{-x}$

2) $f(x) = \sin(ax+b)$ OR $\cos(ax+b)$

3) $f(x) = \sinh ax$ OR $\cosh ax$

4) $f(x) = x^{m}$, m is +ve integer

5) $f(x) = e^{ax} V$, V is any f^n of x

6) $f(x) = x^{m} \sin ax$ OR $x^{m} \cos ax$

7) $f(x) = xV$, V is any f^n of x.

1) $f(x) = e^{ax}$

$$\boxed{\frac{1}{\phi(D)} e^{ax} = \frac{1}{\phi(a)} e^{ax}, \text{ provided } \phi(a) \neq 0}$$

Proof

$\phi(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_n$

$\phi(D) e^{ax} = (a_0 D^n + a_1 D^{n-1} + \cdots + a_n) e^{ax}$

$= a_0 D^n (e^{ax}) + a_1 D^{n-1} (e^{ax}) + \cdots + a_n e^{ax}$

$= a_0 a^n e^{ax} + a_1 a^{n-1} e^{ax} + \cdots + a_n e^{ax}$

$= \phi(a) e^{ax}$

$\frac{1}{\cancel{\phi(D)}} \{ \cancel{\phi(D)} e^{ax} \}$

$= \frac{1}{\phi(D)} \phi(a) e^{ax}$

$\frac{1}{\phi(D)} e^{ax} = \frac{e^{ax}}{\phi(a)}$

Ex:-

1) $\frac{1}{D^2+4} e^{-2x}$

Put $D=-2$

$= \frac{1}{(-2)^2+4} e^{-2x}$

$= \frac{1}{8} e^{-2x}$

2) $\frac{1}{D^2-2D+1} e^{4x}$

Put $D=4$

$= \frac{1}{16-8+1} e^{4x}$

$= \frac{1}{9} e^{4x}$

3) $\frac{1}{D^2+4D+4} e^{-x}$

Put $D=-1$

$= \frac{1}{1-4+4} e^{-x}$

$= e^{-x}$

$$\frac{1}{D^2-4}e^{2x} \qquad \text{deno} = 0.$$

If $\phi(a) = 0$ then

$$\frac{1}{\phi(D)}e^{ax} = x\frac{1}{\phi'(a)}e^{ax}, \text{ provided } \phi'(a) \neq 0$$

eg $$\frac{1}{D^2-4}e^{2x} = x\frac{1}{2D}e^{2x}$$

$$= \frac{x}{4}e^{2x}$$

If $\phi'(a) = 0$ then

$$\frac{1}{\phi(D)} e^{ax} = x^2 \frac{1}{\phi''(a)} e^{ax}, \quad \text{provided } \phi''(a) \neq 0$$

& so on.

eg: $$\frac{1}{D^2 - 2D + 1} e^{x} = x \frac{1}{2D - 2} e^{x} \qquad [\text{because } \phi(1) = 0]$$

$$= x^2 \frac{1}{2} e^{x}$$

$$f(x) = k = K \cdot e^{0x}$$

$$\frac{1}{\phi(D)} K = \frac{1}{\phi(D)} K \cdot e^{0x}$$

$$= \frac{K}{\phi(0)} \quad \text{provided } \phi(0) \neq 0$$

eg: $\frac{1}{D^2+4}(5) \xrightarrow{D=0} \frac{5}{4}.$

$$f(x) = a^x = e^{\log a^x} = e^{x(\log a)}$$

$$\frac{1}{\phi(D)} a^x = \frac{1}{\phi(\log a)} a^x$$

eg $$\frac{1}{D^2+3} 5^x = \frac{1}{(\log 5)^2+3} 5^x$$

$$f(x) = a^{-x}$$

$$\frac{1}{\phi(D)} a^{-x} = \frac{1}{\phi(-\log a)} a^{-x}$$

$$(D^2+2D+1)y = e^{-2x} + 4^x + 3.$$

$$AE: D^2+2D+1 = 0$$

$$D = -1, -1$$

$$y_c = (c_1 x + c_2)e^{-x}$$

$$y_p = \frac{1}{D^2+2D+1}\left(e^{-2x} + 4^x + 3\right)$$

$$= \frac{1}{D^2+2D+1}e^{-2x} + \frac{1}{D^2+2D+1}4^x + \frac{1}{D^2+2D+1}(3)$$

$$= y_{p_1} + y_{p_2} + y_{p_3}$$

$$y_{p_1} = \frac{1}{D^2+2D+1}e^{-2x} \xrightarrow{D=-2} e^{-2x}$$

$$y_{p_2} = \frac{1}{(D+1)^2}4^x \xrightarrow{D=\log 4} \frac{1}{(\log 4+1)^2}4^x$$

$$y_{p_3} = \frac{1}{D^2+2D+1}(3)$$

$$D=0$$

$$= 3$$

$$(D^2+4)y = (e^x+1)^2$$

$$D^2+4=0$$

$$D = \pm 2i$$

$$y_c = c_1\cos 2x + c_2 \sin 2x$$

$$y_p = \frac{1}{D^2+4}\left[e^{2x}+2e^x+1\right]$$

$$= \frac{1}{8}e^{2x} + 2\frac{e^x}{5} + \frac{1}{4}$$