Recall

$$y_p = \frac{1}{\phi(D)} e^{ax} = \frac{1}{\phi(a)} e^{ax}, \quad \phi(a) \neq 0$$

if $\phi(a) = 0$

$$\frac{1}{\phi(D)} e^{ax} = x \frac{1}{\phi'(a)} e^{ax}, \quad \phi'(a) \neq 0$$

$$\frac{1}{\phi(D)} K = \frac{1}{\phi(0)} K$$

$$\frac{1}{\phi(D)} a^x = \frac{1}{\phi(\log a)} a^x$$

$f(x) = \sin(ax+b)$ OR $\cos(ax+b)$

$D \sin(ax+b) = a\cos(ax+b)$

$D^2 \sin(ax+b) = -a^2 \sin(ax+b)$

$$\frac{1}{\phi(D^2)} \sin(ax+b) = \frac{1}{\phi(-a^2)} \sin(ax+b)$$

provided $\phi(-a^2) \neq 0$

$$\frac{1}{\phi(D^2)} \cos(ax+b) = \frac{1}{\phi(-a^2)} \cos(ax+b)$$

Note:-

1) Replace D^2 by $-a^2$
2) Write $D^3 = D^2 \cdot D$, $D^4 = (D^2)^2$, etc
3) Keep D as it is
4) If linear term in D is present in the deno. then rationalise to get D^2.

1) $\sin^2 x = \dfrac{1-\cos 2x}{2}$

2) $\cos^2 x = \dfrac{1+\cos 2x}{2}$

3) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

4) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

5) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

① $\frac{1}{D^2+4}\sin x$

Put $D^2 = -1^2 = -1$

$= \frac{1}{-1+4}\sin x$

$= \frac{1}{3}\sin x$

② $\frac{1}{D^2+2D+1}\cos x$

Put $D^2 = -1$

$= \frac{1}{-1+2D+1}\cos x$

$= \frac{1}{2D}\cos x$

$= \frac{1}{2}\int \cos x\,dx$

$= \frac{1}{2}\sin x$

③ $\frac{1}{D^2+4}\sin 2x$

Put $D^2 = -4$

deno $= 0$

$\therefore\ = x\frac{1}{2D}\sin 2x$

$= \frac{x}{2}\int \sin 2x\,dx$

$= \frac{x}{2}\left(\frac{-\cos 2x}{2}\right)$

④ $\frac{1}{D^3+8}\sin 3x$

$D^2 = -9$

$\therefore D^3 = D^2 \cdot D = -9D$

$= \frac{1}{8-9D}\sin 3x$

$= \frac{8+9D}{64-81D^2}\sin 3x$

$= \frac{8+9D}{64+(81\times 9)}\sin 3x$

$= \frac{1}{793}(8+9D)\sin 3x$

$= \frac{1}{793}(8\sin 3x + 27\cos 3x)$

⑤ $\frac{1}{D^2+1}\sin^2 x$

$= \frac{1}{D^2+1}\left(\frac{1-\cos 2x}{2}\right)$

$= \frac{1}{2}\left[\frac{1}{D^2+1}1 - \frac{1}{D^2+1}\cos 2x\right]$

$D=0 \qquad D^2=-4$

$= \frac{1}{2}\left[1 + \frac{1}{3}\cos 2x\right]$

$$f(x) = \sinh ax \quad \text{OR} \quad \cosh ax$$

$$\sinh ax = \frac{e^{ax} - e^{-ax}}{2}$$

$$\cosh ax = \frac{e^{ax} + e^{-ax}}{2}$$

$$\frac{d}{dx}(\sinh ax) = a\cosh ax$$

$$\frac{d}{dx}(\cosh ax) = a\sinh ax$$

$$\frac{1}{\phi(D^2)} \sinh ax = \frac{1}{\phi(a^2)} \sinh ax$$

provided $\phi(a^2) \neq 0$

$$\frac{1}{\phi(D^2)} \cosh ax = \frac{1}{\phi(a^2)} \cosh ax$$

provided $\phi(a^2) \neq 0$

Replace D^2 by a^2

(& follow rules of sine or cosine)

1) $$\frac{1}{D^2+5}\sinh 2x$$

Put $D^2 = 4$

$$= \frac{1}{4+5}\sinh 2x$$

$$= \frac{1}{9}\sinh 2x$$

2) $$\frac{1}{D^2+2D+1}\cosh x$$

Put $D^2 = 1$

$$= \frac{1}{1+2D+1}\cosh x$$

$$= \frac{1}{2+2D}\cosh x$$

$$= \frac{1-D}{2(1-D^2)}\cosh x$$

$$= \frac{1}{2}(1-D)\left[\frac{1}{1-D^2}\cosh x\right]$$

$$= \frac{1}{2}(1-D)\left[x\frac{1}{-2D}\cosh x\right]$$

$$= \frac{1}{2}(1-D)\left[-\frac{x}{2}\sinh x\right]$$

$$= -\frac{1}{4}\left[x\sinh x - D(x\sinh x)\right]$$

$$= -\frac{1}{4}\left[x\sinh x - (x\cosh x + \sinh x)\right]$$

$f(x) = x^m$, m is +ve integer

$$\boxed{\begin{aligned} \frac{1}{1-z} &= 1+z+z^2+z^3+\cdots \\ \frac{1}{1+z} &= 1-z+z^2-z^3+\cdots \end{aligned}}$$

eg:- ① $\frac{1}{1+\underbrace{D^2}_{z}} x^3$

$= \left[1 - D^2 + (D^2)^2\right] x^3$

$= \left[1 - D^2 + D^4\right] x^3$

$= x^3 - D^2(x^3) + D^4(x^3)$

$= x^3 - 6x + 0$

$= x^3 - 6x$.

② $\frac{1}{D^3+8} x^4$

$= \frac{1}{8\left[1+\left(D^3/8\right)\right]} x^4$

$= \frac{1}{8}\left[1 - \left(D^3/8\right) + \left(D^3/8\right)^2\right] x^4 = \frac{1}{8}\left[x^4 - \frac{1}{8} \times 24x\right]$

$= \frac{1}{8}\left[x^4 - 3x\right]$

1.2

(21) $(D^2+13D+36)y = e^{-4x} + \sinh x$.

A E: $D^2+13D+36=0$

$D = -9, -4$

$\therefore y_c = c_1 e^{-9x} + c_2 e^{-4x}$

$$y_p = \frac{1}{D^2+13D+36} e^{-4x} + \frac{1}{D^2+13D+36} \sinh x$$

$$= y_{p_1} + y_{p_2}.$$

$$y_{p_1} = \frac{1}{D^2+13D+36} e^{-4x}$$ Put $D=-4$

$$= x \frac{1}{2D+13} e^{-4x}$$

$$y_{p_1} = \frac{x}{5} e^{-4x}$$

$$y_{p_2} = \frac{1}{D^2+13D+36} \sinh x$$ Put $D^2=1$

$$= \frac{1}{13D+37} \sinh x = \frac{13D-37}{169D^2-1369} \sinh x$$

$$= \frac{-1}{1200}(13D-37)\sinh x = \frac{-1}{1200}(13\cosh x - 37\sinh x)$$

$\frac{1.2}{22}$ $(D^3+1)y = \sin(2x+3) + e^{-x} + 2^x$

AE: $D^3+1=0$

$D = -1, \frac{1 \pm i\sqrt{3}}{2}$

$y_c = c_1 e^{-x} + e^{x/2}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$

$y_p = \frac{1}{D^3+1}\sin(2x+3) + \frac{1}{D^3+1}e^{-x} + \frac{1}{D^3+1}2^x$

$= y_{p_1} + y_{p_2} + y_{p_3}$

$y_{p_3} = \frac{1}{D^3+1}2^x = \frac{1}{(\log 2)^3+1}2^x$

$y_{p_2} = \frac{1}{D^3+1}e^{-x}$ Put $D=-1$

$= x\frac{1}{3D^2}e^{-x} = \frac{x}{3}e^{-x}$

$y_{p_1} = \frac{1}{D^3+1}\sin(2x+3)$

Put $D^2=-4$ $D^3 = D^2 \cdot D = -4D$

$= \frac{1}{-4D+1}\sin(2x+3)$

$= \frac{1+4D}{1-16D^2}\sin(2x+3)$

$= \frac{1}{65}\left[\sin(2x+3) + 8\cos(2x+3)\right]$

1.2

(19) $(D^3 - 25D)y = \cosh 2x \sinh 3x$

AE $D^3 - 25D = 0$

$D(D^2 - 25) = 0$

$D = 0, 5, -5$

$\therefore y_c = c_1 + c_2 e^{5x} + c_3 e^{-5x}$

$$y_p = \frac{1}{D^3 - 25D} \cosh 2x \sinh 3x$$

$$= \frac{1}{D^3 - 25D} \left[\frac{\sinh 5x + \sinh x}{2} \right]$$

$$= \frac{1}{2} \left[\frac{1}{D^3 - 25D} \sinh 5x + \frac{1}{D^3 - 25D} \sinh x \right]$$

$D^2 = 25$ $\qquad$ $D^2 = 1$

$$= \frac{1}{2} \left[x \frac{1}{3D^2 - 25} \sinh 5x + \frac{1}{D - 25D} \sinh x \right]$$

$$= \frac{1}{2} \left[\frac{x}{50} \sinh 5x - \frac{1}{24D} \sinh x \right]$$

$$y_p = \frac{1}{2} \left[\frac{x}{50} \sinh 5x - \frac{1}{24} \cosh x \right]$$

$f(x) = e^{ax} V$, V is any fn of x

$$D(e^{ax}V) = D(e^{ax})V + e^{ax}DV$$
$$= ae^{ax}V + e^{ax}DV$$
$$= e^{ax}(D+a)V$$

$$D^2(e^{ax}V) = e^{ax}(D+a)^2 V$$

$$\boxed{\frac{1}{\phi(D)} e^{ax} V = e^{ax} \frac{1}{\phi(D+a)} V}$$

ex 1 $(D^2+2D+1)y = e^{-x}\sin 2x$

$$y_p = \frac{1}{D^2+2D+1} e^{-x}\sin 2x$$
$$= e^{-x} \frac{1}{(D-1)^2+2(D-1)+1} \sin 2x$$
$$= e^{-x}\left(\frac{1}{D^2}\sin 2x\right)$$
$$= e^{-x}\left(-\frac{1}{4}\sin 2x\right) \quad (D^2 = -4)$$