PAGE NO.
DATE :

12/04/24
FRI

UNIT : NO : 6 : (SIX) :-
NUMERICAL METHODS.

1.) Interpolation.
   i.) Lagranges Interpolation.
   ii.) Newton's Interpolation.

2.) Numerical Differentiation.

3.) Numerical Integration.
   (1) Trapezoidal Method.
   (2) Simpson's $1/3^{rd}$ Rule.
   (3) Simpson's $3/8^{rd}$ Rule.

Recall :- Algebraic and Transcendental Equations.
Simultaneous Linear Equations.

x	x_0	x_1	x_2	x_3	....	x_n
y	y_0	y_1	y_2	y_3	....	y_n

Approximate $y \cong$ polynomial function.

Value of y for some $x \in (x_0, x_n) \rightarrow$ Process of finding value of y is called 'Interpolation'

- Finite Differences :-

$x_0, x_1, x_2, \ldots\ldots, x_n$ (Equally spaced.)

$x_1 - x_0 = h$, $x_2 - x_1 = h$, ... , $x_n - x_{n-1} = h$

x_0, $x_0 + h$, $x_0 + 2h$, ... $x_0 + nh$ OR

x_0, $\underbrace{x_0 + h}_{= x_1}$, $\underbrace{x + h}_{= x_2}$, $\underbrace{x_2 + h}_{= x_3}$, ... $x_{n-1} + h$

PAGE NO.
DATE :

12/04/24
FRI

UNIT: NO: 6: (SIX):-

NUMERICAL METHODS.

1.) Interpolation.
   i.) Lagranges Interpolation.
   ii.) Newton's Interpolation.

2.) Numerical Differentiation.

3.) Numerical Integration.
   1. Trapezoidal Method.
   2. Simpson's $1/3^{rd}$ Rule.
   3. Simpson's $3/8^{rd}$ Rule.

Recall:- Algebraic and Transcendental Equations.
Simultaneous Linear Equations.

x	x_0	x_1	x_2	x_3	....	x_n
y	y_0	y_1	y_2	y_3	....	y_n

Approximate $y \cong$ polynomial function.

Value of y for some $x \in (x_0, x_n) \rightarrow$ Process of finding value of y is called 'Interpolation'

- Finite Differences:-

$x_0, x_1, x_2, \ldots\ldots, x_n$ (Equally spaced.)

$x_1 - x_0 = h$, $x_2 - x_1 = h$, ..., $x_n - x_{n-1} = h$

x_0, $x_0 + h$, $x_0 + 2h$, ... $x_0 + nh$ OR

x_0, $\underset{x_1}{\underset{\parallel}{x_0 + h}}$, $\underset{x_2}{\underset{\parallel}{x + h}}$, $\underset{x_3}{\underset{\parallel}{x_2 + h}}$, ... $x_{n-1} + h$

x	x_0	x_1	x_2	x_3	....	x_n
y	y_0	y_1	y_2	y_3	....	y_n

Forward Difference (Δ) :-

$$\Delta f(x) = f(x+h) - f(x) \quad \text{.... (First Order Forward Difference.)}$$

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1 \text{ , etc.}$$

$$\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x) \quad \text{........ (second order Forward Difference)}$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0	$y_1 - y_0 = \Delta y_0$	$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$	$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$	$\Delta^3 y_1 - \Delta^3 y_0 = \Delta^4$
x_1	y_1	$y_2 - y_1 = \Delta y_1$	$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$	$\Delta^2 y_2 - \Delta^2 y_1 = \Delta^3 y_1$	
x_2	y_2	$y_3 - y_2 = \Delta y_2$	$\Delta y_3 - \Delta y_2 = \Delta^2 y_2$		
x_3	y_3	$y_4 - y_3 = \Delta y_3$			
x_4	y_4	$y_5 - y_4 = \Delta y_4$			

Ex.] Write the Forward difference table if :-

x	10	20	30	40
y	1.1	2.0	4.4	7.9

ANS. soln :-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
10	1.1 (y_0)	0.9 (Δy_0)	1.5 ($\Delta^2 y_0$)	-0.4 ($\Delta^3 y_0$)
20	2.0	2.4	1.1	
30	4.4	3.5		
40	7.9			

Backward Difference :-

→ $\boxed{\nabla f(x) = f(x) - f(x-h)}$ ...... (First Order Backward Difference)

$\nabla y_1 = y_1 - y_0$

$\nabla y_2 = y_2 - y_1$,... etc.

$\boxed{\nabla^2 f(x) = \nabla f(x) - \nabla f(x-h)}$ ...... (Second Order Backward Difference)

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		$y_1 - y_0 = \nabla y_0$			
x_1	y_1		$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$		
		$y_2 - y_1 = \nabla y_2$		$\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$	
x_2	y_2		$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$		$\nabla^2 y_4 - \nabla^2 y_5 = \nabla^4 y_4$
		$y_3 - y_2 = \nabla y_3$		$\nabla^2 y_4 - \nabla^2 y_3 = \nabla^2 y_4$	
x_3	y_3		$\nabla y_4 - \nabla y_3 = \nabla^2 y_4$		
		$y_4 - y_2 = \nabla y_2$			
x_4	y_4				

Ex.] Construct a backward Difference table if :-

→

x	10	20	30	40
y	1.1	2.0	4.4	7.9

ANS. Soln :-

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
10	1.1			
		0.9		
20	2.0		1.5	
		2.4		-0.4 ($\nabla^3 y_3$)
30	4.4		1.1 ($\nabla^2 y_3$)	
		3.5 (∇y_3)		
40	7.9 (y_3)			

PAGE NO.
DATE :

LAGRANGE'S INTERPOLATION:-

→

x	x_0	x_1	x_2	x_3	....	x_n
y	y_0	y_1	y_2	y_n	....	y_n

... [If sometimes data is not equally spaced.]

$y \cong f(x)$ = polynomial Function (Lagrange's Polynomial)

→ $y \cong L_0(x) y_0 + L_1(x) y_1 + L_2(x) y_2 + \ldots\ldots + L_n(x) y_n$

where,

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)}$$

$$L_1(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}$$

⋮

$$L_n(x) = \frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})}$$

$$y \cong L_0(x) y_0 + L_1(x) y_1 + \ldots + L_n(x) y_n$$

$y(x_0) = y_0 \longrightarrow L_0(x_0) = 1$

$L_1(x_0) = 0, \ldots L_n(x_0) = 0$

Ex1] Use Lagrange's Interpolation formula, to find the value of y when $x = 10$, if the following values of x and y are given:-

→

x	5	6	9	11
y	12	13	14	16

ANS. Soln:- $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$.

$y_0 = 12$, $y_1 = 13$, $y_2 = 14$, $y_3 = 16$.

DATE :

$$y = L_0(x)y + L_1(x)y_1 + L_2(x)y_2 + L_3(x)y_3$$

$$\therefore y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x_2-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

Now, Put $x = 10$ ... (Given)

$$y(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-9)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$\therefore y(10) = \frac{\cancel{4} \times 1 \times (-1)}{(-1)\times(-\cancel{4})\times(-\cancel{6})} \times \cancel{12}^{2} + \frac{\cancel{5} \times 4 \times (\cancel{-1})}{(\cancel{1})(\cancel{-3})(\cancel{-8})} \times 13 + \frac{(5)(\cancel{4})(-1)}{(\cancel{4})(3)(-2)} \times 14$$

$$+ \frac{(\cancel{5})(\cancel{4})^{2}(1)}{(\cancel{6})_{3}(\cancel{5})(\cancel{2})} \times \cancel{16}8$$

$$\therefore y(10) = 2 \; \cancel{7}\;\overset{}{\underset{4}{+\cancel{7}33}} + 11.66 + 5.33 = 14.66$$

$$\boxed{y(10) = 14.66}$$

Q2] The Following table gives the viscosity of an oil as a function of Temperature. Use Lagrange's Formula to find viscosity of oil at a temperature of 140°C.

→

Temp°(t) :	110	150	160	190
Viscosity (V) :	10.8	8.1	5.5	4.8

ANS. Soln:-

Given:- $t_0 = 110$, $t_1 = 150$, $t_2 = 160$, $t_3 = 190$

$V_0 = 10.8$, $V_1 = 8.1$, $V_2 = 5.5$, $V_3 = 4.8$

$$v(t) = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)} \times V_0 + \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)} \times V_1$$

$$+ \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)} \times V_2 + \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_3-t_0)(t_3-t_1)(t_3-t_2)} \times V_3$$

Put, $t = 140\ ^\circ C$ ...(Given)

$$v(140) = \frac{(140-130)(140-160)(140-190)}{(110-130)(110-160)(110-190)} \times 10.8 + \frac{(140-110)(140-160)(140-190)}{(130-110)(130-160)(130-190)} \times 8.1$$
$$+ \frac{(140-110)(140-130)(140-190)}{(160-110)(160-130)(160-190)} \times 5.5 + \frac{(140-110)(140-130)(140-160)}{(190-110)(190-130)(190-160)} \times 4.8$$

$$v(140) = \cancel{\frac{(10)(-20)(-50)}{(-20)(-50)(-70)}} \times 10.8 + \frac{(30)(-20)(-50)}{(20)(-30)(-60)} \times 8.1$$
$$+ \frac{(30)(10)(-50)}{(50)(30)(-30)} \times 5.5 + \frac{(30)(10)(-20)}{(70)(60)(30)} \times 4.8$$

$$v(140) = -\frac{10.8}{7} + \frac{(5 \times 8.1)}{6} + \frac{5.5}{3} - \frac{(2 \times 4.8)}{42}$$

$$v(140) = -1.5428 + 6.75 + 1.833 - 0.2285$$

$$\boxed{v(140) = 6.8117} \quad \underline{\underline{7.03}}$$

Ex 3] Find the Polynomial $F(x)$ by using Lagrange's Formula and hence find $F(3)$ for

→

x	0	1	2	5
$F(x)$	2	3	12	147

ANS. Soln:-

Given:- $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 5$

$y_0 = 2$, $y_1 = 3$, $y_2 = 12$, $y_3 = 147$

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1 +$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$\therefore f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} \times 2 + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} \times 3$$
$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} \times 12 + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} \times 147$$

$$f(x) = -\frac{(x-1)(x-2)(x-5)}{5} + \frac{4x(x-2)(x-5)}{3} - \frac{\cancel{12}^{2}x(x-1)(x-5)}{\cancel{6}}$$

$$+\frac{\cancel{147}^{49}x(x-1)(x-2)}{5\times4\times\cancel{3}}$$

$$f(x) = -(x-1)(x-2)(x-5) + \frac{(4x^2-8x)(x-5)}{3} - (2x^2-2x)(x-5)$$

$$+\frac{49}{20}(x^2-2x-1x+2)$$

$$f(x) = -x^3+8x^2-17x+10 + \frac{4}{3}(x^3-7x^2+10x) - 2x^3+12x^2-10x$$

$$+\frac{49}{20}(x^2-3x+2)$$

$$f(x) = -3x^3+20x^2-27x+10+\frac{4x^3}{3}+\frac{49x^2}{20}-\frac{28x^2}{3}+\frac{40x}{3}-\frac{49\times3x}{20}$$

$$+\frac{49}{10}$$

$$f(x) = -\frac{5}{3}x^3+\frac{787}{60}x^2-\frac{1261x}{60}+\frac{149}{10} =$$

$$f(x) = \frac{-100x^3+787x^2-1261x+894}{60} = \boxed{24.9}$$

$$f(x) = x^3+x^2-x+2 = 35$$

$$f(3) = 27+9-3+2 = 35$$

NEWTON'S FORWARD INTERPOLATION FORMULA :-

1.) Forward difference Operator (Δ)

$$\Delta f(x) = f(x+h) - f(x)$$

2.) Backword difference Operator (∇)

$$\nabla f(x) = f(x) - f(x-h)$$

3.) Shifting Operator (E)

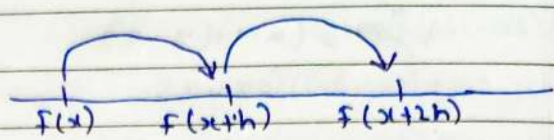
Def^n :- $Ef(x) = f(x+h)$

$$E^2 f(x) = E(Ef(x)) = E(f(x+h)) = f(x+2h)$$

$\vdots$

$$E^n f(x) = f(x+nh)$$

$$E^{-1} f(x) = f(x+nh)$$



$\vdots$

$$E^n f(x) = f(x-nh)$$

4) Relation of Δ and E :-

$\rightarrow$ $\Delta f(x) = f(x+h) - f(x)$

$= Ef(x) - f(x) = (E-1) f(x)$

$$\therefore \boxed{\Delta = E - 1 \quad \text{OR} \quad E = 1 + \Delta}$$

5) Relation of ∇ and E :-

$\rightarrow$ $\nabla f(x) = f(x) - f(x-h)$

$= f(x) - E^{-1} f(x)$

$= (1 - E^{-1}) f(x)$

$$\therefore \boxed{\nabla = 1 - E^{-1}} \quad \text{OR} \quad \boxed{E^{-1} = 1 - \nabla}$$

* NEWTON'S FORWARD INTERPOLATION FORMULA :-

$\rightarrow$

x	:	x_0	x_1	x_2	x_3	x_4	....	x_n ($\rightarrow$ Equally spaced.)
y	:	y_0	y_1	y_2	y_3	y_4	....	y_n

$\rightarrow$ $x_i - x_{i-1} = h$

Find value of y at some x

$x = x_0 + uh$ $\qquad \therefore u = \dfrac{x - x_0}{h}$

$f(x) \cong f(x_0 + uh)$

$= E^u f(x_0)$

$= (1+\Delta)^u \cdot f(x_0) = (1+\Delta)^u \cdot y_0$

$$f(x) = \left[1 + u\Delta + \frac{u(u-1)}{2!}\Delta^2 + \frac{u(u-1)(u-2)}{3!}\Delta^3 + \dots \right] y_0$$

$$\therefore \boxed{f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots}$$

Ex.] Using Newton's Forward Formula, find the value of $f(1.6)$ if

x	1	1.4	1.8	2.2
$f(x)$	3.49	4.82	5.96	6.5

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	3.49 y_0	1.33 Δy_0	-0.19 $\Delta^2 y_0$	-0.41 $\Delta^3 y_0$
1.4	4.82	1.14	-0.6	
1.8	5.96	0.54		
2.2	6.5			

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\cdot\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

Now, $u = \frac{x - x_0}{h} = \frac{1.6 - 1}{0.4} = \frac{0.6}{0.4} = \frac{6}{4} = 1.5$

$$\therefore f(x) = 3.49 + (1.5)(1.33) + \frac{1.5(1.5-1)}{2}(-0.19) + \frac{1.5(1.5-1)(1.5-2)\times(-0.41)}{6}$$

$$\therefore \boxed{f(x) = 5.54}$$

Ex.] Construct Newton's forward interpolation polynomial for the following data.

→

x	4	6	8	10
y	1	3	8	16

Hence, Evaluate y for $x = 5$

ANS. Solution:- $x = ?\ (5)$, $x_n = 10$, $h = 2$

DATE :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1 y_0			
		2 Δy_0		
6	3		3 $\Delta^2 y_0$	
		5		0 $\Delta^3 y_0$
8	8		3	
		8		
10	16			

$$y = f(x) = y_0 + \Delta y_0 \cdot u + \frac{u(u-1)}{2!} \cdot \Delta^2 y_0 + \cdots\cdots$$

Now, $u = \frac{x - x_0}{h} = \frac{x-4}{2}$ $\boxed{u = \frac{x-4}{2}}$

$$\therefore y = f(x) = 1 + \left(\frac{x-4}{2}\right) \times 2 + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2} - 1\right)}{2} \times 3$$

$$\therefore y = 1 + (x-4) + \left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right) \times \frac{3}{2}$$

$$\therefore y = 1 + (x-4) + \frac{3}{8}(x^2 - 10x + 24)$$

$(3/8 \times 24)$

$$\therefore y = \frac{3}{8}x^2 - \frac{30x}{8} + x + \overset{9}{\cancel{24}} + 1 - 4$$

$$\therefore y = \frac{3}{8}x^2 - \frac{22x}{8} + 6$$

$$\boxed{y = \frac{3}{8}x^2 - \frac{22x}{8} + 6}$$

Now, $y(5) = \frac{3 \times 25}{8} - \frac{22 \times 5}{8} + 6 = \frac{13}{8} = 1.625$

$$\boxed{y(5) = 1.625}$$

NEWTON'S BACKWARD INTERPOLATION FORMULA:-

$\rightarrow$ $x = x_n + uh$ $\therefore$ $\boxed{u = \frac{x - x_n}{h}}$

$$\therefore \boxed{f(x) = y_n + u \nabla y_n + \frac{u(u+1)}{2!} \cdot \nabla^2 y_n + \frac{u(u+1)(u+2)}{3!} \cdot \nabla^3 y_n + \cdots}$$

Ex.] Find $f(22)$ from the following data using Newton's backward formulae.

→

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	231	204

ANS. soln :- $x_n = 45$, $x = 22$, $h = 5$

$$u = \frac{x - x_n}{h} = \frac{22 - 45}{5} = -4.6$$

$$\boxed{u = -4.6}$$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
20	354					
		-22				
25	332		-19			
		-41		29		
30	291		10		37	
		-31		-8		-29 ($\nabla^5 y_n$)
35	260		2		8 ($\nabla^4 y_n$)	
		-29		0 ($\nabla^3 y_n$)		
40	231		2 ($\nabla^2 y_n$)			
		-27 (∇y_n)				
45	204 (y_n)					

Now,

$$y = y_n + u\nabla y_n + \frac{u(u+1)\nabla^2 y_n}{2!} + \frac{u(u+1)(u+2)\nabla^3 y_n}{3!} + \frac{u(u+1)(u+2)(u+3)\cdot\nabla^4 y_n}{4!} + \frac{u(u+1)(u+2)(u+3)(u+4)\cdot\nabla^5 y_n}{5!}$$

$$\therefore y = 204 + (-4.6)(-27) + \frac{(-4.6)(-4.6+1)}{2} \times 2 + \frac{(-4.6)(-4.6+1)(-4.6+2)}{\cancel{4}\ 3 \times 2 \times 1} \times 0$$

$$+ \frac{(-4.6)(-4.6+1)(-4.6+2)(-4.6+3)}{4 \times 3 \times 2 \times 1} \times 8 +$$

$$+ \frac{(-4.6)(-4.6+1)(-4.6+2)(-4.6+3)(-4.6+4)}{5 \times 4 \times 3 \times 2 \times 1} \times (-29)$$

$$\therefore \boxed{y = 352}$$

DATE :

NUMERICAL INTEGRATION:-

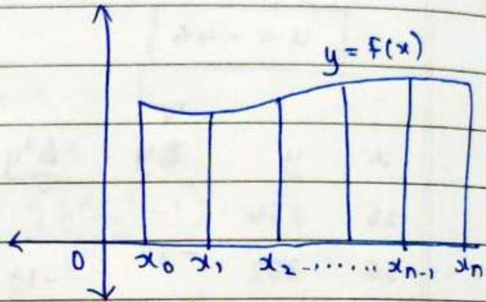
→ Trapezoidal Rule.

Simpson's 1/3rd Rule.

Simpson's 3/8th Rule

x	x_0	x_1	x_2	x_3	…………	x_n
y	y_0	y_1	y_2	y_3	…………	y_n

$I = \int_{x_0}^{x_n} y\,dx$ … (Area under the curve $y=f(x)$ between x_0 to x_n)



$$y \simeq y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

$$\boxed{I = \int_{x_0}^{x_n} y \cdot dx}$$

$x_n = x_0 + nh$

$u = \frac{x - x_0}{h}$ → $du = \frac{1}{h}[dx]$ $\therefore dx = h\,du$

$y \rightarrow f(u) \longrightarrow f^n$ of x

When $x = x_0 \Rightarrow u = \frac{x_0 - x_0}{h} = 0$

$x = x_n \Rightarrow u = \frac{x_n - x_0}{h} = \frac{x_0 + nh - x_0}{h} = \underline{\underline{n}}$

$$\therefore I = \int_0^n \left[y_0 + \Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots\ldots \right] h\,du$$

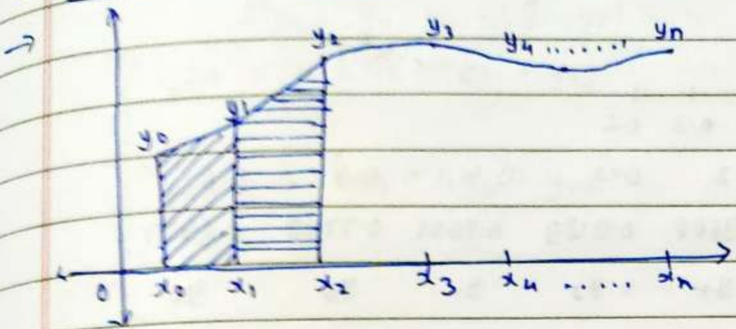
$$\therefore I = h\left[u y_0 + \frac{u^2}{2}\cdot\Delta y_0 + \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Delta^2 y_0 + \ldots\ldots \right]_0^n$$

$$\therefore \boxed{I = h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 n_0 + \ldots \right]} \longrightarrow (1)$$

↳ (Newton's Quotes Formula)

PAGE NO.
DATE :

- **Trapezoidal Rule:-**



$$I = \int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_1} y\,dx + \int_{x_1}^{x_2} y\,dx + \int_{x_2}^{x_3} y\,dx + \ldots\ldots + \int_{x_{n-1}}^{x_n} y\,dx.$$

$$I = \int_{x_0}^{x_1} y\,dx = h\left[(1)\,y_0 + \frac{1}{2}(\Delta y_0)\right] \ldots\ldots \left[\text{Putting } n=1 \text{ in Eqn } \textcircled{1}\right]$$

$$\int_{x_0}^{x_1} y\,dx = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = h\left[y_0 + \frac{1}{2}y_1 - \frac{1}{2}y_0\right] = h\left[\frac{y_0}{2} + \frac{y_1}{2}\right]$$

$$\int_{x_0}^{x_1} y\,dx = \frac{h}{2}\left[y_0 + y_1\right]$$

Similarly, $\int_{x_1}^{x_2} y\,dx = \frac{h}{2}\left[y_1 + y_2\right]$, $\int_{x_2}^{x_3} y\,dx = \frac{h}{2}\left[y_2 + y_3\right]$

$\ldots\ldots \int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}\left[y_{n-1} + y_n\right]$

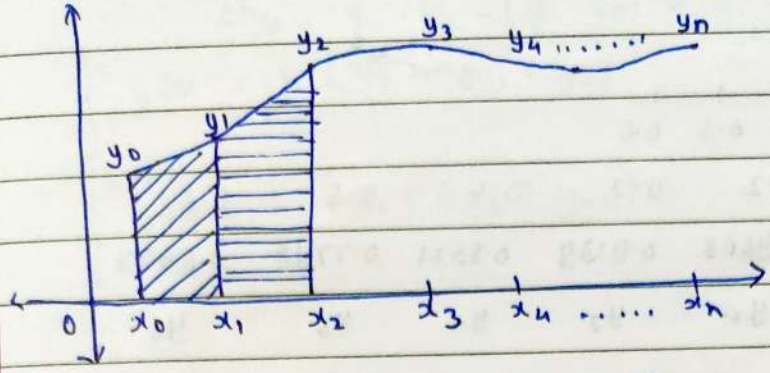
$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0 + y_1\right] + \frac{h}{2}\left[y_1 + y_2\right] + \ldots\ldots + \frac{h}{2}\left[y_{n-1} + y_n\right]$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\left[y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1}\right]\right] \longrightarrow \textcircled{2}$$

This formula (2) is called Trapezoidal Rule.

* Trapezoidal Rule :-

→



$$I = \int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_1} y\,dx + \int_{x_1}^{x_2} y\,dx + \int_{x_2}^{x_3} y\,dx + \ldots\ldots + \int_{x_{n-1}}^{x_n} y\,dx.$$

$$I = \int_{x_0}^{x_1} y\,dx = h\left[(1)\,y_0 + \frac{1}{2}(\Delta y_0)\right] \ldots\ldots \left[\text{Putting } n = 1 \text{ in Eqn } ①\right]$$

$$\int_{x_0}^{x_1} y\,dx = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = h\left[y_0 + \frac{1}{2}y_1 - \frac{1}{2}y_0\right] = h\left[\frac{y_0}{2} + \frac{y_1}{2}\right]$$

$$\int_{x_0}^{x_1} y\,dx = \frac{h}{2}\left[y_0 + y_1\right]$$

Similarly, $\displaystyle\int_{x_1}^{x_2} y\,dx = \frac{h}{2}[y_1 + y_2]$, $\displaystyle\int_{x_2}^{x_3} y\,dx = \frac{h}{2}[y_2 + y_3]$

$\ldots\ldots \displaystyle\int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}[y_{n-1} + y_n]$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[y_0 + y_1] + \frac{h}{2}[y_1 + y_2] + \ldots\ldots + \frac{h}{2}[y_{n-1} + y_n]$$

$$\int_{x_0}^{x_1} y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\left[y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1}\right]\right] \longrightarrow ②$$

This formula ② is called Trapezoidal Rule.

Ex:] Using Trapezoidal Rule, Find $\int_0^{0.6} e^{-x^2} dx$ by taking seven ordinates

$\rightarrow$

$y = e^{-x^2}$

0 0.1 0.2 0.3 0.4 0.5 0.6

x	0	0.1	0.2	0.3	0.4	0.5	0.6
y	1	0.9900	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Now,

$$\int_0^{0.6} e^{-x^2} dx = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{0.1}{2}\left[(1+0.6977) + 2(0.9900+0.9608+0.9139+0.8521+0.7788)\right]$$

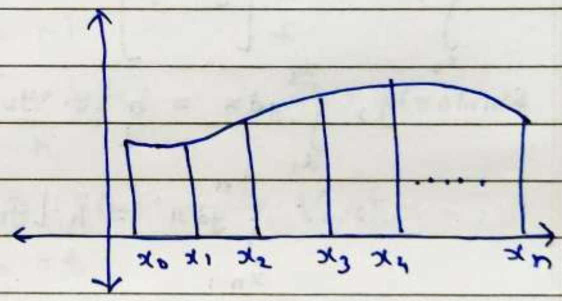
$$= 0.53445$$

$$\boxed{\int_0^{0.6} e^{-x^2} dx = 0.53445}$$

$\Rightarrow$ Simpson's $1/3^{rd}$ Rule :-

$n=2$ Required

Even number intervals.



x_0 y_0 $\rangle\ y_1 - y_0 = \Delta y_0$

x_1 y_1 $\rangle\ \Delta y_1 - \Delta y_0 = \Delta^2 y_0$

x_2 y_2 $\rangle\ y_2 - y_1 = \Delta y_1$

$$y \cong y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\cdot\Delta^2 y_0$$

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_2} y\,dx + \int_{x_2}^{x_4} y\,dx + \int_{x_4}^{x_6} y\,dx + \ldots\ldots + \int_{x_{n-2}}^{x_n} y\,dx$$

Put $n=2$ in ①

$$\int_{x_0}^{x_2} y\,dx = h\left[2y_0 + 2\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - \frac{4}{3}\right)\cdot\Delta^2 y_0\right]$$

DATE :

$\Delta y_0 = y_1 - y_0$

$\Delta^2 y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$$\int_{x_0}^{x_2} y\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

$$\int_{x_2}^{x_4} y\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$$\int_{x_{n-2}}^{x_n} y\,dx = \frac{h}{3}(y_{n-2} + 4\cdot y_{n-1} + y_n)$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \dots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})\right] \longrightarrow (3)$$

This Equation (3) is called Simpson's $\frac{1}{3}$rd Rule.

Ex) Using Simpsons $\frac{1}{3}$rd rule, Evaluate $\int_0^{0.6} e^{-x^2}.dx$ by taking seven ordinates.

→

x	0	0.1	0.2	0.3	0.4	0.5	0.6
y	1	0.9900	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

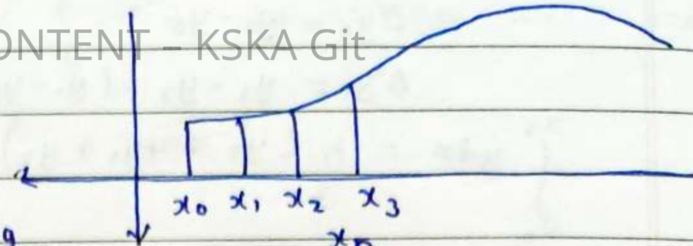
$$\int_0^{0.6} y\,dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{0.1}{3}\left[(1 + 0.6977) + 4(0.9900 + 0.9139 + 0.7788) + 2(0.9608 + 0.8521)\right]$$

$$= 0.53514$$

$$\boxed{\int_0^{0.6} y\,dx = 0.53514}$$

Simpson's $\frac{3}{8}^{th}$ Rule:-



→ $n = 3$

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_3} y\,dx + \int_{x_3}^{x_6} y\,dx + \int_{x_6}^{x_9} y\,dx + \dots + \int_{x_{n-3}}^{x_n} y\,dx.$$

$$= \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 + y_9 + \dots + y_{n-3}) + 3(y_1 + y_2 + \cancel{y_3 + y_4} + y_4 + y_5 + \dots + y_{n-1}) \right]$$

Ex.) **Evaluate :-** $\int_0^6 \frac{dx}{1+x^2}$ by using Simpson's $\frac{3}{8}^{th}$ Rule

→

x	0	1	2	3	4	5	6
y	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

ANS. **soln:-** $h = 1$

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5) \right]$$

$$= \frac{3(1)}{8}\left[(1 + 0.027) + 2(0.1) + 3(0.5 + 0.2 + 0.588 + 0.0385) \right]$$

$$= 1.3571$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.3571$$

FORMULAE:-

(1.) Trapezoidal Rule:-

→ $$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+\cdots\cdots+y_{n-1})\right]$$

(2.) Simpson's $\frac{1}{3}$ rd Rule:-

→ $$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+\cdots+y_{n-2}) + 4(y_1+y_3+\cdots+y_{n-1})\right]$$

(3.) Simpson's $\frac{3}{8}$ th Rule:-

→ $$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+\cdots+y_{n-1}) + 2(y_3+y_6+\cdots\cdots+y_{n-3})\right]$$

Ex.] The Table below shows the Temperature F(t) as a function of time:-

→

t	1	2	3	4	5	6	7
f(t)	81	75	80	83	78	70	60
	f_0	f_1	f_2	f_3	f_4	f_5	f_6

Using Simpson's $\frac{1}{3}$ rd Rule, Estimate $\int_1^7 f(t)\,dt$

ANS. Soln:- $h = 1$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+\cdots+y_{n-2}) + 4(y_1+y_3+\cdots+y_{n-1})\right]$$

$$\therefore \int_1^7 f(t)\,dt = \frac{(1)}{3}\left[(81+60) + 4(75+83+70) + 2(80+78)\right]$$

$$= \frac{1}{3}\left[1369\right] = 456.33$$

Ex.) Use Trapezoidal Rule to evaluate $\int_0^1 x^3\,dx$ considering five subintervals.

$\rightarrow$

0	1/5	2/5	3/5	4/5	1
x_0	x_1	x_2	x_3	x_4	x_5

ANS. Soln:- $h = 1/5$

x	0	1/5	2/5	3/5	4/5	1
$y = x^3$	0	0.008	0.064	0.216	0.512	1
	y_0	y_1	y_2	y_3	y_4	y_5

$$I = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right]$$

$$\therefore I = \frac{1/5}{2}\left[(0+1) + 2(0.008 + 0.064 + 0.216 + 0.512)\right]$$

$$\therefore I = \frac{0.2}{2}\left[1 + 1.6\right] = 0.1(2.6) = 0.26$$

$$\boxed{I = 0.26}$$

For Exact Ans $= \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4} = 0.25$

H.W.

Q.) Evaluate :- $\int_0^1 \frac{dx}{1+x}$ using

1.) Trapezoidal Rule
2.) Simpson's $1/3^{rd}$ Rule.
3.) Simpson's $3/8^{th}$ Rule.

DATE :

Numerical Methods to find the solution of ODE.

→ 1. Euler's Method
2. Euler's Modified Method
3. Runge - Kutta Method of order 4

(1.) **Euler's Method:-**

→ $$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x,y)\,dx$$

→ $$[y]_{y_0}^{y_1} = f(x_0,y_0)\int_{x_0}^{x_1} dx$$

$$\therefore\ y_1 - y_0 = f(x_0,y_0)(x_1 - x_0)$$

$$\boxed{y_1 = y_0 + hf(x_0,y_0)}$$

$$y_2 = y_1 + hF(x_1,y_1)$$

$$\vdots$$

$$\boxed{y_n = y_{n-1} + hF(x_{n-1},y_{n-1})}$$

Q.] Use Euler's Method to solve the differential Equation $\frac{dy}{dx} = x^2 + y$, subject to the condition $x=0, y=1$ $(0(0.1), 0.5)$

ANS. Soln :-

$$f(x,y) = x^2 + y$$

$$x_0 = 0,\ y_0 = 1$$

x	0	0.1	0.2	0.3	0.4	0.5
y	1	1.1	1.21	1.33	1.47	1.66.

$$F(x_0,y_0) = x_0^2 + y = 0 + 1 = 1$$

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + 0.1(1)$$
$$\boxed{y_1 = 1.1.}$$

$f(x_1, y_1) = x_1^2 + y_1 = (0.1)^2 + (1.1) = 1.11$

$y_2 = y_1 + h f(x_1, y_1)$

$\therefore y_2 = 1.1 + (0.1)(1.11)$

$\therefore \boxed{y_2 = 1.21}$

$f(x_2, y_2) = 1.25$

$y_3 = y_2 + h f(x_2, y_2)$

$\therefore y_3 = 1.21 + (0.1)(1.25)$

$\boxed{y_3 = 1.33}$

$f(x_3, y_3) = 1.42$

$y_4 = y_3 + h f(x_3, y_3)$

$\boxed{y_4 = 1.472}$

$\cancel{f(x_4, y_4)} = y_3 + \cancel{h f(x_3, y_3)}$

$=$

$f(x_4, y_4) = x + y$

$= 0.4 + 1.472$

$=$

$=$ calculating ....

$f(x_4, y_4) = 1.872$

$y_5 = y_4 + h f(x_4, y_4)$

$y_5 = 1.472 + (0.1)(1.872)$

$y_5 = 1.6592$

Q] Use Euler's Method to solve the differential Equation $\frac{dy}{dx} = x+y$

, subject to the condition $x = 0$, $y = 0$ $y(0.4)$, $y(0.6)$

ANS. SOLUTION:- $\boxed{h = 0.2}$

x	0	0.2	0.4	0.6
y	0	0	0.04	0.128

$$f(x,y) = x+y$$

$$x_0 = 0,\ y_0 = 0$$

Now, $f(x_0,y_0) = x_0+y_0 = 0+0 = 0$

$$\boxed{f(x_0,y_0) = 0}$$

$$y_1 = y_0 + h \cdot f(x_0,y_0)$$

$$\therefore y_1 = 0 + (0.2)(0) = 0$$

$$\boxed{y_1 = 0}$$

$$f(x_1,y_1) = x_1+y_1$$

$$f(x_1,y_1) = (0.2) + 0 = 0.2$$

$$\boxed{f(x_1,y_1) = 0.2}$$

$$\therefore y_2 = y_1 + h \cdot f(x_1,y_1)$$

$$\therefore y_2 = 0 + (0.2)(0.2)$$

$$\therefore \boxed{y_2 = 0.04}$$

$$f(x_2,y_2) = x_2+y_2$$

$$f(x_2,y_2) = 0.4 + 0.04 = 0.44$$

$$\boxed{f(x_2,y_2) = 0.44}$$

$$\therefore y_3 = y_2 + h \cdot f(x_2,y_2)$$

$$\therefore y_3 = 0.04 + 0.2(0.44) = 0.128$$

$$\boxed{y_3 = 0.128}$$

$$f(x_3,y_3) = x_3+y_3$$

$$\therefore f(x_3,y_3) = 0.6 + 0.128 = 0.728$$

$$\boxed{f(x_3,y_3) = 0.728}$$

$$y_4 = y_3 + h \cdot f(x_3, y_3)$$

$$\therefore y_4 = 0.128 + (0.2 \times 0.728)$$

$$\therefore \boxed{y_4 = 0.2736.}$$

PAGE NO.
DATE :

(3.) EULER'S MODIFIED METHOD:-

→ $$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$\vdots$$

$$y_1^{(n)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right]$$

Q.] Use Euler's Modified Method to solve the differential Equation $\frac{dy}{dx} = x+y$, subject to the condition $x=0, y=0$

$y(0.4), y(0.6)$ $\boxed{h = 0.2}$

$$f(x_0, y_0) = x_0 + y_0 = 0 + 0$$
$$\boxed{f(x_0, y_0) = 0}$$

ANS. soln:-

	x_1, y_1	x_2, y_2	x_3, y_3	x_4, y_4
x	0	0.2	0.4	0.6
y	0	0.22	0.062	0.0662

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$
$$y_1^{(0)} = 0 + (0.2)(0) = 0 \qquad \boxed{y_1^{(0)} = 0}$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$\therefore y_1^{(1)} = 0 + \frac{0.2}{2}\left[0 + (0.2 + 0)\right] = 0.1(0.2) = 0.22$$

$$\boxed{y_1^{(1)} = 0.22}$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$\therefore y_1^{(2)} = 0 + \frac{0.2}{2}\left[0 + (x_1 + y_1^{(1)})\right] = 0 + 0.1\left[0 + 0.4 + 0.22\right]$$
$$= 0.1 \times 0.62 = 0.062$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)}) \right]$$

$$\therefore y_1^{(3)} = 0 + \frac{0.2}{2}\left[0 + (0.6 + 0.062) \right] = 0.1 \times 0.662 = 0.0662$$

$$\boxed{y_1^{(3)} = 0.662}$$